A Fuchs-type Theorem for Partial Differential Equations

by

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1. Introduction

A well-known theorem of Fuchs establishes the existence of solutions to "regular singular" ordinary differential equations which have at worst logarithmic singularities or poles (see [In]). Under certain conditions on the roots of the so-called indicial equation these solutions may be analytic, for example, the Bessel equation which has entire solutions (Bessel functions of the first kind). Baouendi and Goulaouic pioneered a theorem of this type for partial differential equations in [BG]. Also, see subsequent work of Igari [Ig].

The theorem in this paper, although obtained independently, contains virtually the same results. Our method, however, has perhaps two important differences. First, using a lemma of Hörmander and adapting his proof of the Cauchy-Kovalevskaya theorem to the characteristic case makes our proof more straightforward and elementary then those in [BG] and [Ig]. Also, using the globalizing family method (cf. [J1,2], [K], [KS]), for particular operators we have extended the solution to a sufficiently large domain, in contrast to the local solution in [BG].

We are considering differential operators of the following form:

$$\mathcal{L} := y^m D_y^m + \sum_{k=0}^{m-1} y^k b^k D_y^k + \sum_{k=1}^{m} \sum_{|\alpha|+j=k} y^k a^{\alpha,j} D_x^\alpha D_y^j \quad , \quad |\alpha| \geq 1$$

where we are using the standard multi-index notation for $x = (x_1, x_2, \cdots, x_N) \in \mathbb{C}^N$, $(x, y) \in \mathbb{C}^{N+1}$:

$$\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_N) \in \mathbb{N}^N \, , \quad |\alpha| = \sum_{i=1}^{N} \alpha_i \, , \quad D_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1{}^{\alpha_1} \partial x_2{}^{\alpha_2} \cdots \partial x_N{}^{\alpha_N}} \, , \quad D_y^j = \frac{\partial^j}{\partial y^j} \, ,$$

and b^k, $a^{\alpha,j}$ are holomorphic in $\bar{D}(0, 1) := \{x \in \mathbb{C}^N \colon |x_j| \leq 1, \ j = 1, 2, ..., N\}$.

We give a sufficient condition, which is an extension of the conditions on the roots of the indicial equation in the ordinary differential equation case, under which the characteristic Cauchy problem:

$$\begin{cases} \mathcal{L}u = 0 \\ D_y^j u(x, 0) = v_j \quad , \qquad 0 \leq j \leq m-1 \end{cases} \tag{1}$$

has an analytic solution, where v_j is analytic in $\bar{D}(0, 1)$ for $0 \leq j \leq m - 1$.

2. Theorem

Theorem: The characteristic Cauchy problem (1) has an analytic solution in a neighborhood of the origin, provided

$$\mathrm{P}(x,\, m,\, n) := n(n-1)\cdots(n-m+1) + n(n-1)\cdots(n-m+2)\, b_0^{m\text{-}1}(x) + \cdots + n\, b_0^1(x) + b_0^0(x) \neq 0$$

for all $n \geq m$ and $x \in \bar{D}(0,\, 1)$, and $v_0 b_0^0(x) = 0$, i. e. $v_0(x) = 0$, or $b_0^0(x) = 0$.

Proof: We are seeking a solution of (1) in the form of a power series

$$u(x,\, y) = \sum_{n=0}^{\infty} u_n(x)\, y^n.$$

Let $a^{\alpha,j}(x,\, y) = \sum_{n=0}^{\infty} a_n^{\alpha,j}(x)\, y^n$, $b^k(x,\, y) = \sum_{n=0}^{\infty} b_n^k(x)\, y^n$. All coefficients in x are assumed to be analytic in $\bar{D}(0,\, 1)$.

Putting these into the equation we have

$$\sum_{n=m}^{\infty} u_n\, n(n-1)\cdots(n-m+1) y^n + \sum_{k=0}^{m-1} \left\{ \sum_{n=k}^{\infty} \left(\sum_{s=0}^{n} b_{n-s}^k\, u_s\, s(s-1)\cdots(s-k+1) \right) y^n \right\} +$$

$$+ \sum_{k=1}^{m} \left\{ \sum_{|\alpha|+j=k}\ \sum_{n=k}^{\infty} \left(\sum_{s=0}^{n-k} a_{n-k-s}^{\alpha,j}\, D_x^{\alpha}\, u_{s+j}\, (s+j)(s+j-1)\cdots(s+1) \right) y^n \right\} = 0.$$

Therefore, we have the following recurrence relation for the coefficients

$$u_n = -\frac{1}{P(x,\, m,\, n)} \left(\sum_{k=0}^{m-1} \sum_{s=0}^{n-1} b_{n-s}^k\, u_s\, s(s-1)\cdots(s-k+1) + \right.$$

$$\left. + \sum_{k=1}^{m}\ \sum_{|\alpha|+j=k}\ \sum_{s=0}^{n-k} a_{n-k-s}^{\alpha,j}\, D_x^{\alpha}\, u_{s+j}\, (s+j)(s+j-1)\cdots(s+1) \right) \tag{2}$$

for all $n \geq m$.

It suffices to show

$$|u_n(x)| \le \frac{C^n}{[(1 - |x_1|)(1 - |x_2|)\cdots(1 - |x_N|)]^n}, \tag{3}$$

for $n \in \mathbf{N}$, where C is a constant to be specified later.

Then,

$$\sum_{n=0}^{\infty} |u_n(x)y^n| \le \sum_{n=0}^{\infty} \frac{C^n}{[(1 - |x_1|)(1 - |x_2|)\cdots(1 - |x_N|)]^n} |y|^n$$

converges in the neighborhood $|x_i| < 1 - \frac{1}{K}$, $|y| < \frac{1}{CK^N}$ for all $K \in \mathbf{N}$.

Since each v_j is holomorphic in $\bar{D}(0, 1)$ for $0 \le j \le m - 1$, by (1), there exists a constant C such that

$$|u_j(x)| \le \frac{C^j}{[(1 - |x_1|)(1 - |x_2|)\cdots(1 - |x_N|)]^j}, \quad \text{for } 0 \le j \le m - 1. \tag{4}$$

To show (3) for $n \ge m$, we need the following (see [H]):

<u>Lemma</u>: If $f(z)$ is analytic in $\mathbf{D} = \{z \in \mathbf{C}: |z| < 1\}$ and

$$|f(z)| \le \frac{1}{(1 - |z|)^k},$$

then,

$$|f'(z)| \le \frac{(k + 1)\, e}{(1 - |z|)^{k+1}}.$$

From the condition on $P(x, m, n)$ there exists a constant $d > 0$ such that

$$|P(x, m, n)| \ge dn^m \text{ for all } n \in \mathbf{N}. \tag{5}$$

Thus, by (2) and (5) we have

$$|u_n| \le \frac{1}{dn^m}\left(\sum_{k=0}^{m-1} \sum_{s=0}^{n-1} |b_{n-s}^k|\, |u_s|\, s(s-1)\cdots(s-k+1) + \right.$$
$$\left. + \sum_{k=1}^{m} \sum_{|\alpha|+j=k} \sum_{s=0}^{n-k} |a_{n-k-s}^{\alpha,j}|\, |D_x^\alpha u_{s+j}|\, (s+j)(s+j-1)\cdots(s+1) \right).$$

Assume (3) holds for all $n < m$, then the lemma gives us

$$|u_n| \le \frac{1}{dn^m}\left(\sum_{k=0}^{m-1} \sum_{s=0}^{n-1} |b_{n-s}^k| \frac{\mathrm{C}^s\, s(s-1)\cdots(s-k+1)}{[(1-|x_1|)(1-|x_2|)\cdots(1-|x_\mathrm{N}|)]^s} + \right.$$

$$\left. + \sum_{k=1}^{m} \sum_{|\alpha|+j=k} \sum_{s=0}^{n-k} |a_{n-k-s}^{\alpha,j}| \frac{\mathrm{C}^{s+j}\, e^{|\alpha|}\, \mathrm{K}(s,\, j,\, \alpha)}{[(1-|x_1|)(1-|x_2|)\cdots(1-|x_\mathrm{N}|)]^{s+j}(1-|x_1|)^{\alpha_1}(1-|x_2|)^{\alpha_2}\cdots(1-|x_\mathrm{N}|)^{\alpha_\mathrm{N}}} \right),$$

where $\mathrm{K}(s,\, j,\, \alpha) = (s+j+\alpha_\mathrm{N})\cdots(s+j+1)\cdots(s+j+\alpha_1)\cdots(s+j+1)(s+j)(s+j-1)\cdots(s+1)$.

Therefore,

$$|u_n| \le \frac{\mathrm{C}^n}{[(1-|x_1|)(1-|x_2|)\cdots(1-|x_\mathrm{N}|)]^n}\left(\frac{1}{d\mathrm{C}} \sum_{k=0}^{m-1} \mathrm{B}^k + \frac{e^m}{d\mathrm{C}} \sum_{k=1}^{m} \sum_{|\alpha|+j=k} \mathrm{A}^{\alpha,j} \right)$$

where B^k and $\mathrm{A}^{\alpha,j}$ satisfy $|b_n^k| \le \mathrm{B}^k$ and $|a_n^{\alpha,j}| \le \mathrm{A}^{\alpha,j}$ for each $n \in \mathbf{N}$.

Choosing C so that, in addition to (4),

$$\left(\frac{1}{d\mathrm{C}} \sum_{k=0}^{m-1} \mathrm{B}^k + \frac{e^m}{d\mathrm{C}} \sum_{k=1}^{m} \sum_{|\alpha|+j=k} \mathrm{A}^{\alpha,j} \right) \le 1$$

completes the proof.

Remark: We chose $y^k \ a^{\alpha,j} \ D_x^\alpha \ D_y^j$, where $k = |\alpha| + j$ for the mixed derivative terms in our differential operator. In our proof, as in [BG] there is flexibility on the exponent on y. Namely, if the operator is of the form

$$\mathcal{L} := y^s \ D_y^m + \sum_{k=1}^{s} \ y^{s-k} \ b^{m-k} \ D_y^{m-k} \ + \sum_{k=0}^{m-s-1} \ b^k \ D_y^k + \sum_{k=1}^{m} \ \sum_{|\alpha|+j=k} \ y^{\mathrm{Q}} \ a^{\alpha,j} \ D_x^\alpha \ D_y^j \ \text{, where } s \leq m,$$

we may choose $\mathrm{Q} = \max(0,\ s + j - m + 1)$. We can then use the same method to solve the Cauchy problem (1) for this operator, provided

$$\mathrm{P}(x,\ m,\ n) := n(n-1)\cdots(n-m+1) + n(n-1)\cdots(n-m+2)b_0^{m-1}(x) + \cdots + nb_0^{m-s+1}(x) + b_0^{m-s}(x) \neq 0,$$

and the v_j satisfy the corresponding compatibility condition determined by $\mathcal{L}u(x,\ 0) = 0$, as in the following example.

3. Globalizing Family Method (an example)

The globalizing family method allows us to extend our solution for certain operators to a domain whose size depends on the principal symbol. For further discussion of these notions see [J1,2], [K], [KS]. We will now consider, as an example, the Cauchy problem for $(x,\ y) \in \mathbb{C}^2$:

$$\mathcal{L}u := y\Delta u + a(x,\ y)\ u_x + b^1(x,\ y)\ u_y + b^0(x,\ y)\ u = 0, \quad u(x,\ 0) = f(x), \quad u_y(x,\ 0) = g(x), \qquad (6)$$

where $\mathrm{P}(x,\ 2,\ n) = n(n-1) + nb_0^1(x) \neq 0$, $n \geq 2$, and f and g satisfy $a_0 f'(x) + b_0^1 g(x) + b_0^0 f(x) = 0$.

Let $\mathbf{D}_x := \{x \in \mathbb{C}\colon |x| < 1\}$ and $\hat{\mathbf{D}}$ be the Lie ball: $\hat{\mathbf{D}} := \{(x,y) \in \mathbb{C}^2\colon |x \pm iy| < 1\}$. Let $\mathrm{V}_1 \subset \mathrm{V}_2$ be bounded open sets in $\mathbb{C}^2$. A globalizing family with respect to V_1, V_2 and $\mathcal{L}$, is a family $\{\Omega_t\}_{t \in [0,\ 1]}$ of smoothly bounded open subsets of V_2 such that:

(i) $\Omega_{t_1} \subset \Omega_{t_2}$, for $t_1 < t_2$;

(ii) $\exists \delta > 0$ such that $\Omega_t \subset \mathrm{V}_1$ as $t < \delta$;

(iii) $\Omega_{t_0} = \bigcup_{t<t_0} \Omega_t$ and $\bar{\Omega}_{t_0} = \bigcap_{t>t_0} \Omega_t$;

(iv) for all $t \in [0, 1]$ and each $z_0 \in \partial\Omega_t \backslash V_1$, $\partial\Omega_t$ is non-characteristic (with respect to $\mathfrak{L}$) at z_0;

(v) $V_2 = \left(\bigcup_t \Omega_t\right) \cup V_1$.

Zerner's theorem and the theorem of Bony-Schapira imply (cf. [KS], [K, ch.8]) the globalizing principle, namely, that if u extends holomorphically to a convex open set V_1 in $\mathbf{C}^2$, where it satisfies $\mathfrak{L}u = F$, where F is holomorphic in a convex domain V_2, then there is a globalizing family $\{\Omega_t\}$ filling out V_2 that allow us to extend u to all of V_2, provided any real hyper-plane $\{(x, y) \in \mathbf{C}^2\colon \operatorname{Re}(x \pm iy) = t\}$ that is characteristic with respect to $\mathfrak{L}$ that intersects V_2 also intersects V_1. We shall apply this to the case where V_1 is a convex neighborhood of $\mathbf{D}_x$ and V_2 is $\hat{\mathbf{D}}$. Note that the only characteristic points on the boundaries of the Ω_t with respect to the operator in (6) are the points in $\partial\Omega_t \cap \{y = 0\}$. Thus, by our theorem, the solution of (6) extends holomorphically into a neighborhood of $\mathbf{D}_x$ in $\mathbf{C}^2$, and the globalizing principle allows us to extend the solution automatically to all of $\hat{\mathbf{D}}$.

4. Concluding remarks

(1) The following example shows the necessity of the condition on $P(x, m, n)$. For the characteristic Cauchy problem

$$y\Delta u - u_y = 0, \; u_0 = f, \; u_1 = 0$$

where $P(x, 2, n) = n(n-1) - n$, then $P(x, 2, 2) = 0$ and if $f'' \neq 0$, we can not find a holomorphic solution in any neighborhood of $\{y = 0\}$.

(2) Our interest in this problem stems from a conjecture in [EKS] on the existence of holomorphic solutions to equations of the type in our example with the Laplacian in the principal part. As stated, the conjecture is not true unless extra conditions are met. However, the operator

$$M = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\alpha + \beta + 1)\frac{\partial}{\partial x} + \frac{\alpha - \beta + (\alpha + \beta + 1)\cos y}{\sin y}\frac{\partial}{\partial y}, \; \alpha > -1$$

needed for the proof of Szegö's theorem in that paper satisfies our condition since $P(x, 2, n) = n(n-1) + n(2\alpha + 1) = n(n + 2\alpha) > 0$ for $n \geq 2$.

(3) In [BG] the uniqueness of the solution is also demonstrated. Uniqueness follows immediately from our proof since we have an explicit recurrence relationship for the coefficients.

(4) The proof of the theorem mimics that of the Cauchy-Kovalevskaya theorem, adapted to having a characteristic problem. Ideally, we would like a proof of this theorem that does not resort to power series expansions. It seems the way to achieve this is to return to the ordinary differential equation situation and find a "high-ground" proof of Fuchs' theorem.

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