Another Look at Some Index Theorems for The Shift

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ANOTHER LOOK AT SOME INDEX THEOREMS FOR THE SHIFT

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ABSTRACT. Letting μ be a finite, positive Borel measure with support in $\{z : |z| \leq 1\}$ such that $\mu(\{z : |z| = 1\}) > 0$, $P^2(\mu)$ (the closure of the polynomials in $L^2(\mu)$) is irreducible and $\{z : |z| < 1\} = abpe(P^2(\mu))$ (the collection of analytic bounded point evaluations for $P^2(\mu)$), we give a condition that is sufficient to ensure that $dim(\mathcal{M} \ominus z\mathcal{M}) = 1$ for each nontrivial closed invariant subspace $\mathcal{M}$ for the shift M_z on $P^2(\mu)$. This condition is possibly a consequence of our general assumptions concerning μ, yet, whether or not it is remains an open question.

1. INTRODUCTION

For $1 \leq t < \infty$ and any finite, positive Borel measure μ with compact support in the complex plane $\mathbf{C}$, we let $P^t(\mu)$ denote the closure of the polynomials in $L^t(\mu)$ and let $abpe(P^t(\mu))$ denote the set of analytic bounded point evaluations for $P^t(\mu)$. In this general setting, J. Thomson has given a direct sum decomposition of $P^t(\mu)$ that involves the components of $abpe(P^t(\mu))$ (see [T]). Multiplication by the independent variable z is a bounded operator on $P^t(\mu)$; we call this operator the shift and denote it by M_z. In this paper we focus on the Hilbert space setting (i.e., $P^2(\mu)$) and we assume that $P^2(\mu)$ is irreducible and that $abpe(P^2(\mu))$ is the unit disk $\mathbf{D} := \{z : |z| < 1\}$; by

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[T], these assumptions imply that the support of μ is contained in $\overline{\mathbf{D}}$ and that $\mu|_{\partial\mathbf{D}} \ll m$, where m denotes normalized Lebesgue measure on $\partial\mathbf{D}$. Two important examples that fall under this heading are given by the cases: $\mu = m$ and $\mu = A$, where A denotes area measure on $\mathbf{D}$. Indeed, $P^2(m)$ can be identified with the Hardy space $H^2(\mathbf{D}) := \{f : f \text{ is analytic in } \mathbf{D} \text{ and } \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty\}$ and $P^2(A)$ with the Bergman space $L^2_a(\mathbf{D}) := \{f : f \text{ is analytic in } \mathbf{D}$ and $\int_{\mathbf{D}} |f|^2 dA < \infty\}$. Beurling's Theorem thoroughly describes the lattice of closed invariant subspaces for M_z on $H^2(\mathbf{D})$. By this description, $dim(\mathcal{M} \ominus z\mathcal{M}) = 1$ for each nontrivial member $\mathcal{M}$ of this lattice. In contrast, the lattice of closed invariant subspaces for M_z on $L^2_a(\mathbf{D})$, which is not well-understood at this point, is very large. An indication of this is found in work of C. Apostol, H. Bercovici, C. Foias and C. Pearcy (see [ABFP]) who have shown (in general) that if $P^2(\mu)$ is irreducible, $abpe(P^2(\mu)) = \mathbf{D}$ and $\mu(\partial\mathbf{D}) = 0$, then, for any natural number n, and for $n = \infty$, there is a closed invariant subspace $\mathcal{M}$ for M_z on $P^2(\mu)$ such that $dim(\mathcal{M} \ominus z\mathcal{M}) = n$; see [HRS] for related work. If, however, $\mu = A + m|_E$, where E is a closed subset of $\partial\mathbf{D}$ that has positive Lebesgue measure and satisfies the Carleson condition, then $P^2(\mu)$ is irreducible and $dim(\mathcal{M} \ominus z\mathcal{M}) = 1$ for every nontrivial, closed invariant subpace $\mathcal{M}$ for M_z on $P^2(\mu)$ (see [M] and [Y]; and see [H] or [KM] for conditions for irreducibility). J. B. Conway and L. Yang have recently conjectured that the outcome is the same in the setting of any irreducible $P^2(\mu)$ space for which $abpe(P^2(\mu)) = \mathbf{D}$ and $\mu(\partial\mathbf{D}) > 0$ (see [CY], Question 2.1). Quite related to this conjecture is some work of R. Olin and J. Thomson (see [OT], Theorem 1) who have shown that if ν is a finite, positive Borel measure with support in $\overline{\mathbf{D}}$ such that $abpe(P^2(\nu)) = \mathbf{D}$, $P^2(\nu)$ is irreducible and the support of ν has an "outer hole", then $dim(\mathcal{M} \ominus z\mathcal{M}) = 1$ for every nontrivial closed invariant subspace $\mathcal{M}$ for M_z on $P^2(\nu)$; J. Thomson and L. Yang have extended this result to the setting of $P^t(\mu)$, for $1 < t < \infty$ (see [TY]). So, a natural strategy for establishing the above conjecture would be to show that if $P^2(\mu)$ is irreducible, $abpe(P^2(\mu)) = \mathbf{D}$ and $\mu(\partial\mathbf{D}) > 0$, then there is a measure μ_o such that the support of μ_o has an outer hole and the shifts on $P^2(\mu)$ and $P^2(\mu_o)$ are similar. In this paper we do not make the full journey in this strategy from hypothesis to conclusion, but describe an intermediate point and move from that to the conclusion. Specifically, if $P^2(\mu)$ is irreducible, $abpe(P^2(\mu)) = \mathbf{D}$ and $\mu(\partial\mathbf{D}) > 0$, there is precedent (see [K]; or [H], Section 0.4) to believe that there is a Jordan curve Γ in $\overline{\mathbf{D}}$ ($\Omega := inside(\Gamma)$) that has the properties:

i) $\omega_\Omega|_{\partial\mathbf{D}} \ll \mu|_{\partial\mathbf{D}}$ (ω_Ω denotes harmonic measure on Γ for evaluation at some point z_o in Ω).

ii) $\omega_\Omega(\partial\mathbf{D}) > 0$.

iii) There exists h, $0 \le h \in L^\infty(\omega_\Omega)$, such that $log(h) \in L^1(\omega_\Omega)$ and $\int_\Gamma |p|^2 h d\omega_\Omega \le \int |p|^2 d\mu$ for all polynomials p.

This idea, along with Lemma 2.3, suggests a condition (see Definition 2.4) that we show is sufficient for one to be able assert the existence of a measure μ_o as described above (Theorem 2.5). We then use Theorem 2.5 (more precisely, Corollary 2.6) to establish the special cases that have appeared in the literature (Corollary 2.7). After this, in Section 3, we basically rework Theorem 1 of [OT], using some methods that shorten its proof (Theorem 3.3). We end the paper with a remark concerning the converse of Theorem 2.5.

2. A Sufficient Condition

Our first result in this section is well-known. Among other references that could be cited here, it is an easy consequence of [P], Proposition 6.23 or results and remarks found in [BCGJ]. Throughout this paper we let m denote normalized Lebesgue measure on $\partial\mathbf{D}$ and, for any bounded, simply connected region G in the complex plane, we let ω_G denote harmonic measure on ∂G for evaluation at some point z_o in G.

Lemma 2.1. *Let G be a simply connected subregion of $\mathbf{D}$ such that $0 \in G$. If E is a Borel subset of $\partial\mathbf{D}$ such that $\omega_G(E) > 0$, then there is a Jordan subregion U of G such that $0 \in U$, ∂U is rectifiable and $\omega_U(E) > 0$.*

If g is a Nevanlinna class function in $\mathbf{D}$, then g has well-defined nontangential boundary values a.e. m; we follow convention and let $\tilde{g}$ denote these boundary values.

Proposition 2.2. *Let Ω be a Jordan subregion of $\mathbf{D}$ such that $0 \in \Omega$ and $\omega_\Omega(\partial\mathbf{D}) > 0$. Let φ be a conformal mapping from $\mathbf{D}$ onto Ω (with $\varphi(0) = 0$), let F be an outer function in $\mathbf{D}$ ($F \not\equiv 0$) and define f on Ω by $f(z) = F(\varphi^{-1}(z))$. Then there is a Jordan subregion U of Ω that contains 0 and has rectifiable boundary, and there exists $\varepsilon > 0$ such that:*

1) $\omega_U(\partial\mathbf{D}) > 0$, and

2)$|f(z)| > \varepsilon$ for all z in U.

Proof. By Lemma 2.1 and a standard conformal mapping argument, we need only show that if $E \subseteq \partial\mathbf{D}$ and $m(E) > 0$, then there is a simply connected subregion G of $\mathbf{D}$ that contains 0 and there exists $\varepsilon > 0$ such that :

i) $\omega_G(E) > 0$, and

ii) $|F(z)| > \varepsilon$ for all z in G.

In pursuit of this, we observe that since $m(E) > 0$ and $F \not\equiv 0$, there exists $c > 0$ such that $m(\{e^{i\theta} \in E : |\tilde{F}(e^{i\theta})| \geq c\}) > 0$. Multiplying F by $\frac{1}{c}$ and making $|\tilde{F}|$ smaller off $\{e^{i\theta} \in E : |\tilde{F}(e^{i\theta})| \geq c\}$ if need be, we reduce this problem to the case that $|\tilde{F}(e^{i\theta})| \geq 1$ for m-almost all $e^{i\theta}$ in E and $|\tilde{F}(e^{i\theta})| \leq \frac{1}{2}$ for m-almost all $e^{i\theta}$ in $(\partial\mathbf{D}) \setminus E$. For $n = 1, 2, 3, \ldots$, let G_n be the component of $\{z \in \mathbf{D} : |F(z)| > \frac{1}{n}\}$ that contains 0; notice that G_n is open, simply connected and nonempty for sufficiently large n. Choose ζ, ζ in E, such that $\tilde{F}(\zeta)$ exists and $|\tilde{F}(\zeta)| \geq 1$. Then there exists δ, $0 < \delta < 1$, such that $|F(r\zeta)| \geq \delta$ for $0 \leq r < 1$ and hence $\{r\zeta : 0 \leq r < 1\} \subseteq G_n$, provided $n > \frac{1}{\delta}$. Select r_o, $0 \leq r_o < 1$, such that $|F(r_o\zeta)| \geq \frac{3}{4}$ and (for some fixed $n > \frac{1}{\delta}$) let ω_n denote harmonic measure on ∂G_n for evaluation at $r_o\zeta$. Now $|F(z)| = \frac{1}{n}$ for all z in $\mathbf{D} \cap (\partial G_n)$ and $|\tilde{F}(e^{i\theta})| \leq \frac{1}{2}$ a.e. m on $(\partial\mathbf{D}) \setminus E$. Therefore, since

$$\begin{aligned} log(\tfrac{3}{4}) &\leq log|F(r_o\zeta)| \\ &= \int_{\mathbf{D}\cap(\partial G_n)} log|F|d\omega_n + \int_{(\partial\mathbf{D})\cap(\partial G_n)} log|\tilde{F}|d\omega_n \\ &\leq \omega_n((\partial G_n) \setminus E) \cdot log(\tfrac{1}{2}) + \int_E log|\tilde{F}|d\omega_n, \end{aligned}$$

and ω_n is a probability measure, we conclude that $\omega_n(E) > 0$. By Harnack's Inequality, $\omega_{G_n}(E) > 0$, where ω_{G_n} denotes harmonic measure on ∂G_n for evaluation at some (arbitrary) point of G_n; and our proof is complete.□

Let μ be a finite, positive Borel measure with support in $\overline{\mathbf{D}}$ and let $P^2(\mu)$ denote the closure of the polynomials in $L^2(\mu)$. We further assume that $\mathbf{D} = abpe(P^2(\mu))$ (the collection of analytic bounded point evaluations for $P^2(\mu)$); [C] and [T] are excellent references for such sets of point evaluations. Then, for each z in $\mathbf{D}$, there exists a unique k_z in $P^2(\mu)$ such that $p(z) = \int p(\zeta)\overline{k_z(\zeta)}d\mu(\zeta)$ for all polynomials p; $||k_z||_{L^2(\mu)} = \sup\{|p(z)| : p \text{ is a polynomial and } ||p||_{L^2(\mu)} = 1\}$ and $z \longrightarrow ||k_z||_{L^2(\mu)}$ is continuous on $\mathbf{D}$. As a result, each f in $P^2(\mu)$ has a natural analytic continuation to $\mathbf{D}$.

Lemma 2.3. *Let μ be a finite, positive Borel measure with support in $\overline{\mathbf{D}}$ such that $abpe(P^2(\mu)) = \mathbf{D}$, and for each z in $\mathbf{D}$, let k_z be the unique (kernel) function in $P^2(\mu)$ (as described above) for evaluation at z. Let ν be any finite, positive Borel measure with support in $\overline{\mathbf{D}}$ such that $\nu(\partial\mathbf{D}) = 0$, and define η on $\mathbf{D}$ by $d\eta(z) = \frac{1}{||k_z||^2_{L^2(\mu)}} \cdot d\nu(z)$. Then*

$$||p||_{L^2(\eta)} \leq \sqrt{\nu(\mathbf{D})} \cdot ||p||_{L^2(\mu)}$$

for all polynomials p.

Proof. For any polynomial p,

$$\begin{aligned} ||p||_{L^2(\eta)} &= \left\{\int_{\mathbf{D}} |p(z)|^2 d\eta(z)\right\}^{\frac{1}{2}} \\ &\leq \left\{\int_{\mathbf{D}} ||p||^2_{L^2(\mu)} \cdot ||k_z||^2_{L^2(\mu)} d\eta(z)\right\}^{\frac{1}{2}} \\ &= \sqrt{\nu(\mathbf{D})} \cdot ||p||_{L^2(\mu)}.\square \end{aligned}$$

Our next result (the main result of this section) is set in the general context we described earlier. Namely, we assume that μ is a finite, positive Borel measure with support in $\overline{\mathbf{D}}$ such that $abpe(P^2(\mu)) = \mathbf{D}$, that $P^2(\mu)$ is irreducible and that $\mu(\partial\mathbf{D}) > 0$; by [T], these assumptions imply that $\mu|_{\partial\mathbf{D}} \ll m$. As in our earlier discussion, for z in $\mathbf{D}$, we let k_z denote the (kernel) function in $P^2(\mu)$ for evaluation at z.

Definition 2.4. With μ as described above, we say that μ is *strongly inscribed* if there is a Jordan subregion Ω of $\mathbf{D}$ ($\Gamma := \partial\Omega$ need not be rectifiable) such that:

1) $\omega_\Omega(\partial\mathbf{D}) > 0$,

2) $\omega_\Omega|_{\partial\mathbf{D}} \ll \mu|_{\partial\mathbf{D}}$ and $\int_{\Gamma\cap\partial\mathbf{D}} log(1 + \frac{d\omega_\Omega}{d\mu})d\omega_\Omega < \infty$, and

3) $\int_{\mathbf{D}\cap\Gamma} log(||k_z||_{L^2(\mu)})d\omega_\Omega(z) < \infty$.

Theorem 2.5. *Let μ be strongly inscribed. Then there is a finite, positive Borel measure μ_o with support in $\overline{\mathbf{D}}$ and with the properties:*

1) There is a constant $M > 1$ such that

$$\frac{1}{M} \cdot ||p||_{L^2(\mu)} \leq ||p||_{L^2(\mu_o)} \leq M \cdot ||p||_{L^2(\mu)}$$

for all polynomials p, and

2) $\mathbf{D} \setminus support(\mu_o)$ contains a Jordan region U with rectifiable boundary such that $0 \in U$ and $\omega_U(\partial\mathbf{D}) > 0$; in the terminology of R. Olin and J. Thomson ([OT]), $support(\mu_o)$ has an "outer hole".

Proof. Since μ is strongly inscribed, there is a Jordan subregion Ω of $\mathbf{D}$ ($\Gamma := \partial\Omega$) having the properties listed in Definition 2.4. In fact, since $z \longrightarrow ||k_z||_{L^2(\mu)}$ is continuous on $\mathbf{D}$, we may assume that $0 \in \Omega$. Let σ denote the sweep of $\mu|_\Omega$ to Γ; by [C], Proposition 9.21, we know that $\sigma \ll \omega_\Omega$. Furthermore, by [OY], Lemma 2.6, we may assume that $\mu(\{0\}) = 1$, and so $\omega_\Omega \leq \sigma$; we let ω_Ω denote harmonic measure on Γ

for evaluation at 0. Now define h on Γ (a.e. ω_Ω) by:

$$\begin{aligned} h(z) &= \frac{1}{||\sigma|| \cdot ||k_z||^2_{L^2(\mu)}} \text{ for } z \text{ in } \Gamma \cap \mathbf{D}, \text{ and} \\ h(z) &= min(1, \frac{d\mu}{d\sigma}(z)) \text{ if } z \in \Gamma \cap (\partial \mathbf{D}). \end{aligned}$$

By our hypotheses and Lemma 2.3, we have:

a) $h \in L^\infty(\omega_\Omega)$,

b) $\int_\Gamma log(h) d\omega_\Omega > -\infty$, and

c) $\int_\Gamma |p|^2 h d\sigma \leq\ 2 \cdot \int |p|^2 d\mu$,

for all polynomials p. Let φ be a conformal mapping from $\mathbf{D}$ onto Ω (since Ω is a Jordan region, φ extends to a homeomorphism between $\overline{\mathbf{D}}$ and $\overline{\Omega}$), let F be an outer function in $\mathbf{D}$ such that $|\tilde{F}(e^{i\theta})| = h(\varphi(e^{i\theta}))$ for m-almost all $e^{i\theta}$ in $\partial\mathbf{D}$ and define f on Ω by $f(z) = F(\varphi^{-1}(z))$; likewise, we define $\tilde{f}$ a.e. ω_Ω on Γ by $\tilde{f}(\zeta) = \tilde{F}(\varphi^{-1}(\zeta))$. Now, by Proposition 2.2, there is a Jordan subregion U of Ω that contains 0 and that has rectifiable boundary, and there exists $\varepsilon > 0$ such that $\omega_U(\partial\mathbf{D}) > 0$ and $|f(z)| > \varepsilon$ for z in U. We let σ_U denote the sweep of $\mu|_U$ to Γ; since $U \subseteq \Omega$, we have: $\sigma_U \leq \sigma$. If p is any polynomial, then, by the subharmonicity of $|p|^2|f|$ on Ω and the fact that $|f| > \varepsilon$ on U,

$$\begin{aligned} \varepsilon \cdot \int_U |p|^2 d\mu \leq \int_U |p|^2 |f| d\mu &\leq \int_\Gamma |p|^2 |\tilde{f}| d\sigma_U \\ &\leq \int_\Gamma |p|^2 |\tilde{f}| d\sigma \\ &= \int_\Gamma |p|^2 h d\sigma \\ &\leq 2 \cdot \int |p|^2 d\mu. \end{aligned}$$

So, if we define μ_o on $K := \overline{\mathbf{D}} \setminus U$ by: $d\mu_o = d\mu|_K + hd\sigma$, then there is a constant $M > 1$ such that

$$\frac{1}{M} \cdot ||p||_{L^2(\mu)} \leq ||p||_{L^2(\mu_o)} \leq M \cdot ||p||_{L^2(\mu)},$$

for all polynomials p.□

Corollary 2.6. *If μ is strongly inscribed, then $dim(\mathcal{M} \ominus z\mathcal{M}) = 1$ for any nontrivial closed invariant subspace $\mathcal{M}$ for the shift M_z on $P^2(\mu)$.*

Proof. Let μ_o be the measure provided by Theorem 2.5 (for μ). By Theorem 2.5, the shifts on $P^2(\mu)$ and $P^2(\mu_o)$ are similar (as operators). Since M_z on $P^2(\mu)$ is pure, so is M_z on $P^2(\mu_o)$ ([C], Chapter II, Proposition 13.11). We can now apply [OT], Theorem 1 to get that $dim(\mathcal{M} \ominus z\mathcal{M}) = 1$ for each nontrivial closed invariant subspace $\mathcal{M}$ for M_z on $P^2(\mu_o)$. By the similarity of the shifts on $P^2(\mu)$ and $P^2(\mu_o)$, our proof is complete.□

We now use Corollary 2.6 to rework an example that appears in the literature (see [Y], Section 2). Recall that a closed subset E of $\partial\mathbf{D}$ is said to satisfy the *Carleson condition* if the intervals $\{I_n\}$ that are complementary to E in $\partial\mathbf{D}$ have the property:

$$\sum_n m(I_n) log(\frac{1}{m(I_n)}) < \infty.$$

Corollary 2.7. *Let E be a proper, closed subset of $\partial\mathbf{D}$ that has positive Lebesgue measure and that satisfies the Carleson condition. Define μ on $\overline{\mathbf{D}}$ by: $\mu = A + m|_E$, where A denotes area measure on $\mathbf{D}$. Then $dim(\mathcal{M} \ominus z\mathcal{M}) = 1$ for each nontrivial closed invariant subspace $\mathcal{M}$ for M_z on $P^2(\mu)$.*

Proof. Let $\{I_n\}$ be the intervals complementary to E in $\partial\mathbf{D}$; we let a_n and b_n denote the endpoints of I_n. By adjoining at most seven points to E (if necessary), we may assume that $m(I_n) \leq \frac{1}{8}$ for all n. For each n, let γ_n be the chord of $\partial\mathbf{D}$ that has endpoints a_n and b_n. Let $\Gamma = (\cup_n \gamma_n) \cup E$ – observe that Γ is a Jordan curve – and let $\Omega = inside(\Gamma)$. Now, for z in $\mathbf{D}$ and any polynomial p,

$$p(z) = \frac{1}{\pi(1-|z|)^2} \cdot \int_{\Delta_z} p(\zeta) dA(\zeta);$$

$\Delta_z := \{\zeta : |z-\zeta| < 1-|z|\}$. Therefore,

$$\begin{aligned} |p(z)| &\leq \frac{1}{\pi(1-|z|)^2} \cdot \left\{\int_{\Delta_z} |p(\zeta)|^2 dA(\zeta)\right\}^{\frac{1}{2}} \\ &\leq \frac{1}{\pi(1-|z|)^2} \cdot ||p||_{L^2(\mu)}, \end{aligned}$$

and so $||k_z||_{L^2(\mu)} \leq \frac{1}{\pi(1-|z|)^2}$. Also, for any chord γ_n and for any z in $\gamma_n \backslash \{a_n, b_n\}$, $1-|z| \geq \frac{1}{8} \cdot |b_n - a_n||z-a_n||z-b_n|$. Moreover, by a standard argument involving the Maximum Principle, there is a positive constant c such that $d\omega_\Omega \leq c ds$; s denotes arclength measure on Γ. So, there is a positive constant C such that

$$\int_{\gamma_n} log||k_z||_{L^2(\mu)} d\omega_\Omega(z) \leq C|b_n - a_n|(3 + log(\frac{1}{|b_n - a_n|})),$$

for all n. Therefore, since E satisfies the Carleson condition,

$$\int_{\Gamma\cap\mathbf{D}} log||k_z||_{L^2(\mu)} d\omega_\Omega(z) < \infty.$$

Now, by the Maximum Principle and the fact that $\Gamma \cap \partial\mathbf{D} = E$, we have: $\omega_\Omega|_{\partial\mathbf{D}} \leq m|_E$ $(=\mu|_E)$. In fact, by the geometry of Ω and [P], Proposition 6.23, $m|_E \ll \omega_\Omega|_E$. Clearly $abpe(P^2(\mu)) = \mathbf{D}$, and indeed, since $m|_E \ll \omega_\Omega|_E$, we can argue as in the proof of Theorem 2.5 to get that $P^2(\mu)$ is irreducible (which in this case means that no nontrivial part of $L^2(m|_E)$ splits as an L^2-summand of $P^2(\mu)$). It now follows that μ is strongly inscribed and so, by Corollary 2.6, we have our conclusion. □

Question 2.8. If μ is a finite, positive Borel measure with support in $\overline{\mathbf{D}}$ such that $\mu(\partial\mathbf{D}) > 0$, $abpe(P^2(\mu)) = \mathbf{D}$ and $P^2(\mu)$ is irreducible, then is μ strongly inscribed?

3. An Index Theorem Revisited

In this section we establish a result (Theorem 3.2) that can be used to shorten the proof of [OT], Theorem 1. We then illustrate the use of Theorem 3.2 in the proof of Theorem 3.3 – a basic form of [OT], Theorem 1. We end this section (and the paper) with a remark concerning the converse of Theorem 2.5

Lemma 3.1. *Let W be a simply connected subregion of $\mathbf{D}$ such that $0 \in W$ and $\omega_W(\partial\mathbf{D}) > 0$; we let $B = \mathbf{D} \setminus W$. Then there is a Jordan subregion V of W (that contains 0 and has rectifiable boundary) and there is a positive constant M such that:*
1) $\omega_V(\partial\mathbf{D}) > 0$, and
2) $\sup_{z\in B}\left|\frac{z\bar{\zeta}-1}{z-\zeta}\right| < M$ for each ζ in V.

Proof. For $n = 1, 2, 3, \ldots$, let $W_n = \{\zeta \in W : \sup_{z\in B}\left|\frac{z\bar{\zeta}-1}{z-\zeta}\right| < n\}$ and let V_n denote the component of W_n that contains 0. Notice that V_n is open, simply connected and nonempty for sufficiently large n. Let $T(W)$ denote the set of tangent points of ∂W with $\partial\mathbf{D}$ (see [BCGJ]); we likewise define $T(V_n)$ to be the set of tangent points of ∂V_n with $\partial\mathbf{D}$. Choose λ in $T(W)$ and let γ be a rectifiable arc in $W \cup \{\lambda\}$ that has endpoints 0 and λ and that has nontangential approach in

W to λ. Notice that there exists n such that $\gamma \setminus \{\lambda\} \subseteq V_n$ and indeed, for this n, $\lambda \in T(V_n)$. Consequently, $T(W) \subseteq \cup_{n=1}^{\infty} T(V_n)$. Now since $\omega_W(\partial \mathbf{D}) > 0$, by [BCGJ], $m(T(W)) > 0$. So there exists n such that $m(T(V_n)) > 0$. Deferring once again to [BCGJ], we conclude that $\omega_{V_n}(\partial \mathbf{D}) > 0$ (for some n). Applying Lemma 2.1 to such a V_n, our proof is complete.□

Let ν be a finite, complex Borel measure with compact support in the complex plane. Then the Cauchy transform of ν, denoted $\hat{\nu}$, is defined and analytic off *support*(ν) and and is given by:

$$\hat{\nu}(\zeta) = \int \frac{d\nu(z)}{z - \zeta}$$

Our next result is somewhat analogous to [TY], Lemma 2.1.

Theorem 3.2. *Let W be a simply connected subregion of $\mathbf{D}$ such that $0 \in W$ and $\omega_W(\partial \mathbf{D}) > 0$. Then there is a Jordan subregion V of W (that contains 0 and has rectifiable boundary) such that:*
1) $\omega_V(\partial \mathbf{D}) > 0$.
2) If ν is any finite, complex Borel measure having support in $\overline{\mathbf{D}} \setminus W$ such that $\int \overline{z}^n d\nu(z) = 0$ for $n = 1, 2, 3, \ldots$, then $\hat{\nu} \in H^1(V)$.

Proof. Let V and M be the Jordan subregion of W and the constant that are given by Lemma 3.1. Now, since $\int \overline{z}^n d\nu(z) = 0$ (for $n = 1, 2, 3, \ldots$), if $\zeta \in V$, then

$$\begin{aligned}
\hat{\nu}(\zeta) &= \int \left(\frac{1}{z - \zeta} - \frac{\overline{z}}{1 - \overline{z}\zeta} \right) d\nu(z) \\
&= \int \frac{1 - |z|^2}{(z - \zeta)(1 - \overline{z}\zeta)} d\nu(z) \\
&= \int_{\mathbf{D}} \frac{1 - |z|^2}{(z - \zeta)(1 - \overline{z}\zeta)} d\nu(z).
\end{aligned}$$

Notice that, for ζ in V and z in $\overline{\mathbf{D}} \setminus W$,

$$\begin{aligned}\psi_z(\zeta) &:= Re\left\{\frac{1+\overline{z}\zeta}{1-\overline{z}\zeta}\right\} \\ &= \frac{1-|z|^2|\zeta|^2}{|1-\overline{z}\zeta|^2} \quad \left(\geq \frac{1-|z|^2}{|1-\overline{z}\zeta|^2}\right)\end{aligned}$$

and $\psi_z(\zeta)$ is harmonic as a function of ζ in V. So, for ζ in V,

$$\begin{aligned}|\hat{\nu}(\zeta)| &\leq \int_{\mathbf{D}} \frac{1-|z|^2}{|z-\zeta||1-\overline{z}\zeta|} d|\nu|(z) \\ &\leq M \cdot \int_{\mathbf{D}} \frac{1-|z|^2}{|1-\overline{z}\zeta|^2} d|\nu|(z) \\ &\leq M \cdot \int_{\mathbf{D}} \psi_z(\zeta) d|\nu|(z).\end{aligned}$$

Evidently, $\psi(\zeta) := M \cdot \int_{\mathbf{D}} \psi_z(\zeta) d|\nu|(z)$ is a harmonic majorant for $|\hat{\nu}|$ on V and so $\hat{\nu} \in H^1(V)$.□

Theorem 3.3. *Let μ_o be the measure given by Theorem 2.5. If $\mathcal{M}$ is a nontrivial closed invariant subspace for the shift M_z on $P^2(\mu_o)$, then $dim(\mathcal{M} \ominus z\mathcal{M}) = 1$.*

Proof. Let $\mathcal{M}$ be a nontrivial closed invariant subspace for M_z on $P^2(\mu_o)$. Now $dim(\mathcal{M} \ominus z\mathcal{M}) \geq 1$, since $abpe(P^2(\mu_o)) = \mathbf{D}$. Suppose that $dim(\mathcal{M} \ominus z\mathcal{M}) \geq 2$; we look for a contradiction. Then we can find f and g in $\mathcal{M} \ominus z\mathcal{M}$ such that $||f||_{L^2(\mu_o)} = ||g||_{L^2(\mu_o)} = 1$ and $\int f\overline{g}d\mu_o = 0$. Indeed, since f and g are in $\mathcal{M} \ominus z\mathcal{M}$, we also have:

1) $\int z^n f\overline{g}d\mu_o = \int \overline{z}^n f\overline{g}d\mu_o = 0$ (for $n = 0, 1, 2, ...$); and likewise with $\overline{f}g$ in place of $f\overline{g}$.

2) $\int z^n |f|^2 d\mu_o = \int \overline{z}^n |f|^2 d\mu_o = 0$ (for $n = 1, 2, 3, \ldots$); and likewise with $|g|^2$ in place of $|f|^2$.

For convenience of notation, if $\varphi \in L^1(\mu_o)$, then we let $\hat{\varphi}$ denote the Cauchy transform of the measure given by $\varphi d\mu_o$. By (1), (2) and Theorem 3.2, there is a Jordan subregion V of U (U is the Jordan region with rectifiable boundary that is given by Theorem 2.5) such that $0 \in V$, ∂V is rectifiable, $\omega_V(\partial \mathbf{D}) > 0$, and such that $(f\overline{g})\hat{}$, $(\overline{f}g)\hat{}$, $(|f|^2)\hat{}$ and $(|g|^2)\hat{}$ are each in $H^1(V)$. Moreover, $(zf\overline{g})\hat{}(\zeta) = \zeta \cdot (f\overline{g})\hat{}(\zeta)$, $(z\overline{f}g)\hat{}(\zeta) = \zeta \cdot (\overline{f}g)\hat{}(\zeta)$, and by expanding in power series in a small disk about 0, we also see that, for ζ in V, $(z|f|^2)\hat{}(\zeta) = 1 + \zeta \cdot (|f|^2)\hat{}(\zeta)$ and $(z|g|^2)\hat{}(\zeta) = 1 + \zeta \cdot (|g|^2)\hat{}(\zeta)$. Therefore $(zf\overline{g})\hat{}$, $(z\overline{f}g)\hat{}$, $(z|f|^2)\hat{}$ and $(z|g|^2)\hat{}$ are each in $H^1(V)$. At this point we recall a simple, yet ingenious observation of R. Olin and J. Thomson in the proof of [OT], Theorem 1. Notice that, by [OT], Lemma 6, $\varphi_1 := (zf\overline{g})\hat{} \cdot (z\overline{f}g)\hat{}$ and $\varphi_2 := (z|f|^2)\hat{} \cdot (z|g|^2)\hat{}$ have the same boundary values a.e. ω_V on $(\partial \mathbf{D}) \cap (\partial V)$, and yet $\varphi_1(0) = 0$ while $\varphi_2(0) = 1$. So, $\varphi := \varphi_2 - \varphi_1$ is in $H^{\frac{1}{2}}(V)$, $\varphi(0) = 1$, and φ has zero boundary values on a set of positive ω_V measure; an obvious contradiction. Therefore, the proof is complete. □

Remark 3.4. It seems likely that the converse of Theorem 2.5 also holds; in what follows we assume the terminology of Theorem 2.5. For if there is a finite, positive Borel measure ν with compact support K in $\mathbf{D}$ (we may assume that K is connected) and there exists g in $L^2(\mu_o + \nu)$ such that $g \perp \{1, z, \overline{z}, z^2, \overline{z}^2, \ldots\}$ and yet $\hat{g}(\zeta) := \int \frac{g(z)d(\mu_o+\nu)(z)}{z-\zeta}$ is not idntically zero on a component G of $U \backslash K$ (where $\omega_G(\partial \mathbf{D}) > 0$), then $p(\zeta) = \frac{1}{\hat{g}(\zeta)} \cdot \int \frac{p(z)g(z)d(\mu_o+\nu)(z)}{z-\zeta}$ for each polynomial p and each ζ in G

for which $\hat{g}(\zeta) \neq 0$. Hence, for such ζ, and by [OY], Lemma 2.6,

$$\begin{aligned} |p(\zeta)| &\leq \frac{1}{|\hat{g}(\zeta)| \cdot dist(\zeta, \partial G)} \cdot ||p||_{L^2(\mu_o+\nu)} \cdot ||g||_{L^2(\mu_o+\nu)} \\ &\leq \frac{const.}{|\hat{g}(\zeta)| \cdot dist(\zeta, \partial G)} ||p||_{L^2(\mu)}. \end{aligned}$$

And so, for such ζ, $||k_\zeta||_{L^2(\mu)} \leq \frac{const.}{|\hat{g}(\zeta)| \cdot dist(\zeta, \partial G)}$. Applying Theorem 3.2 and a variant of [A], Proposition 4, a Jordan subregion Ω of G that satisfies the requirements of Definition 2.4 is now within reach, provided $(\partial G) \cap (\partial \mathbf{D})$ contains a closed set of positive Lebesgue measure that satisfies the Carleson condition.

The question concerning the existence of a measure ν (as described in Remark 3.4) is basically a harmonic polynomial approximation problem and is unresolved at this point.

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