Bounded Composition Operators with Closed Range on the Dirichlet Space

by

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BOUNDED COMPOSITION OPERATORS WITH CLOSED RANGE ON THE DIRICHLET SPACE

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ABSTRACT. For composition operators on spaces of analytic functions it is well known that norm estimates can be converted to Carleson measure estimates. The boundedness of the composition operator becomes equivalent to a Carleson measure inequality. The measure corresponding to a composition operator C_φ on the Dirichet space $\mathcal{D}$ is the $d\nu_\varphi = n_\varphi\,dA$, where $n_\varphi(z)$ is the cardinality of the preimage $\varphi^{-1}(z)$. The composition operator will have closed range if and only if the corresponding measure satisfies a "reverse Carleson measure" theorem: $\|f\|^2_{\mathcal{D}} \le \int |f'|^2\,d\nu_\varphi$ for all $f \in \mathcal{D}$. Assuming C_φ is bounded, a necessary condition for this inequality is a reverse of the Carleson condition: (C) $\nu_\varphi(S) \ge c|S|$ for all Carleson squares S. It has long been known that this is not sufficient for a completely general measure. Here we show that it is also not sufficient for the special measures ν_φ. That is, we construct a function φ such that C_φ is bounded and ν_φ satisfies (C) but the composition operator C_φ does not have closed range.

1. INTRODUCTION

Let φ be an analytic map from the unit disk $\mathbf{D}$ into itself. For any analytic function f on $\mathbf{D}$ define $C_\varphi(f) = f \circ \varphi$. This defines a linear transformation known as a composition operator. Let dA be area measure on the unit disk and let $\mathcal{D}$ denote the *Dirichlet space* of analytic functions on $\mathbf{D}$ satisfying $\int_{\mathbf{D}} |f'|^2\,dA < \infty$. We give $\mathcal{D}$ the norm $\|f\|^2_{\mathcal{D}} = |f(0)|^2 + \int |f'|^2\,dA$. There is a well know necessary and sufficient condition for C_φ to be bounded on $\mathcal{D}$. It arises from the following observation:

$$\int_{\mathbf{D}} |(f\circ\varphi)'|^2\,dA = \int_{\mathbf{D}} |f'(\varphi)|^2|\varphi'|^2\,dA = \int |f'|^2 n_\varphi\,dA$$

where $n_\varphi(w)$ is the cardinality of the set $\varphi^{-1}(w)$. The boundedness of C_φ is equivalent to $n_\varphi\,dA$ being a Carleson measure for the Bergman space. Let $\psi(z,w) = |z-w|/|1-\bar{z}w|$ denote the pseudohyperbolic metric, and for $0<\eta<1$, let $D_\eta(w) = \{z \in \mathbf{D} : \psi(z,w) < \eta\}$. The condition for $n_\varphi\,dA$ to be a Carleson measure for the Bergman space is that there exist a constant C independent of z such that for all (some) $0<\eta<1$

$$\int_{D_\eta(z)} n_\varphi\,dA \le C|D_\eta(z)| \quad \text{for all } z \in \mathbf{D}. \tag{1.1}$$

Here $|D_\eta(z)|$ denotes the area of $D_\eta(z)$.

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For any $\zeta \in \partial\mathbf{D}$ and $0 < h < 2$ let $S(\zeta, h) = \{z \in \mathbf{D} : |z - \zeta| < h\}$. Equivalent to (1.1) is the following condition:

$$\int_{S(\zeta,h)} n_\varphi \, dA \le C|S(\zeta, h)| \quad \text{for all } \zeta \in \partial\mathbf{D} \text{ and all } 0 < h < 2. \tag{1.2}$$

Of course, the constants C in (1.1) and (1.2) need not be the same.

We are concerned with the condition that $C_\varphi : \mathcal{D} \to \mathcal{D}$ have closed range as well as being bounded. After the usual accomodation for the $|f(0)|^2$ term in $\|f\|_{\mathcal{D}}^2$ (see [2]), the necessary and sufficient condition for this is that

$$\frac{1}{C}\int |f'|^2 \, dA \le \int |f'|^2 n_\varphi \, dA \le C \int |f'|^2 \, dA \tag{1.3}$$

for some constant C and all $f \in \mathcal{D}$. A necessary condition for this (see [3]) is condition (1.1) plus the following:

$$\text{For some } 0 < R < 1 \text{ and } \delta > 0 \quad \int_{D_R(z)} n_\varphi \, dA \ge \delta |D_R(z)| \quad \text{for all } z \in \mathbf{D}, \tag{1.4}$$

or the equivalent version

$$\text{For some } \delta > 0 \quad \int_{S(\zeta,h)} n_\varphi \, dA \ge \delta |S(\zeta, h)| \quad \text{for all } S(\zeta, h). \tag{1.5}$$

I and others have conjectured that this necessary condition might be sufficient. The condition in [3] on general Carleson measures (on which condition (1.4) is based) is not in general a sufficient condition for that setting. However, the best counterexamples of that sufficiency were discrete measures concentrated on a zero set for the Bergman space and for a while it seemed possible that measures of the form $n_\varphi dA$ might behave unlike discrete measures. Then the results in [4] indicated perhaps the opposite, but I found it difficult to construct functions φ to make $n_\varphi \, dA$ have the required properties. The examples constructed in [7] and expanded on in [2] provided the tools necessary to produce a counterexample. The rest of this paper is involved in constructing an analytic map φ from $\mathbf{D}$ to itself which satisfies conditions (1.1) and (1.4) but C_φ does not have closed range. Thus, conditions (1.1) and (1.4) are not sufficient for C_φ to have closed range. Section 2 presents some of the main ideas about Carleson measures that will be needed. Lemmas 2.2 and 2.4 isolate the goals of the construction. Section 3 contains the actual construction of φ.

2. Preliminaries on Carleson measures

The Bergman space A^p is defined for any $p > 0$ as the space of all analytic functions on $\mathbf{D}$ such that the 'norm'

$$\|f\|_p = \left(\int_{\mathbf{D}} |f|^p \, dA\right)^{1/p}$$

is finite. A Carleson measure for A^p is a positive measure μ on $\mathbf{D}$ satisfying

$$\int |f|^p \, d\mu \le C\|f\|_p^p \quad \text{for all } f \in A^p. \tag{2.1}$$

The general form of condition (1.1) provides a necessary and sufficient condition for a measure μ to be a Carleson measure:

$$\|\mu\|_* \stackrel{\text{def}}{=} \sup_{z\in\mathbf{D}} \frac{\mu(D_\eta(z))}{|D_\eta(z)|} < \infty. \tag{2.2}$$

Note that we do not indicate the dependency on η in the definition of the norm $\|\mu\|_*$. Different values of η produce equivalent norms, and the value of η will always be fixed within any discussion involving this norm.

If μ is a Carleson measure for A^p then the identity operator is continuous from A^p to $L^p(\mu)$. The condition for this operator to be an isomorphic embedding is

$$\text{For some constant } C \quad \int_{\mathbf{D}} |f|^p\,dA \le C\int |f|^p\,d\mu \quad \text{for all } f\in A^p. \tag{2.3}$$

In [3] it was shown that in the presence of condition (2.2), a necessary condition for (2.3) is

$$\text{For some } 0<R<1 \text{ and } \delta>0 \quad \mu(D_R(z)) > \delta|D_R(z)| \quad \text{for all } z\in\mathbf{D}. \tag{2.4}$$

This is the source of condition (1.4): the set of derivatives of functions in $\mathcal{D}$ is A^2 and the measure in question is $d\mu = n_\varphi\,dA$.

An example showing that (2.4) is not sufficient is easily obtained by constructing and '(η, R)-lattice' which is a zero set for A^p. An (η, R)-lattice is a sequence $\{z_k\}$ in $\mathbf{D}$ such that the disks $D_\eta(z_k)$ are disjoint and the disks $D_R(z_k)$ cover $\mathbf{D}$. Such a sequence is a zero sequence for a function in A^p provided η is sufficiently near 1 (see [1], [5] and [6]). Construct a measure μ to have mass $|D_\eta(z_k)|$ concentrated at each point z_k. It is easily seen that (2.2) is satisfied and (2.4) is satisfied with $\delta = \inf(|D_\eta(z_k)|/|D_R(w)|) > 0$, the infimum being taken over all k and all $w\in D_R(z_k)$. However, (2.3) fails for the functions f having zeros at all z_k. We will later show that a measure need only be concentrated "near" a zero set in an appropriate sense, for (2.3) to fail

A measure ν is called a vanishing Carleson measure if for every $0<\eta<1$

$$\sup_{r<|z|<1} \frac{\nu(D_\eta(z))}{|D_\eta(z)|} \to 0 \quad \text{as } r\to 1.$$

These are the measures such that the identity map from A^p to $L^p(\nu)$ is compact.

In [3] it was pointed out that the implication (2.2) $\Rightarrow$ (2.1) could be improved to the following using the same proof:

Lemma 2.1. *Let μ be a positive measure on $\mathbf{D}$ that satisfies condition* (2.2). *Let E be any Borel set in $\mathbf{D}$ and E_η denote the set of points with pseudohyperbolic distance from E less than η (same η as in* (2.2)). *There is an constant C depending only on η such that if f is analytic in $\mathbf{D}$*

$$\int_E |f|^p\,d\mu \le C\|\mu|_E\|_* \int_{E_\eta} |f|^p\,dA. \tag{2.5}$$

Two things to note about (2.5): the constant C is independent of μ and the integral on the right could be taken on quite a small subset of $\mathbf{D}$. Note that a vanishing Carleson measure ν is one satisfying $\|\nu|_{A_r}\|_* \to 0$ as $r\to 1$, where A_r denotes the annulus $r<|z|<1$. This observation makes the following almost obvious.

Lemma 2.2. *If ν is a vanishing Carleson measure and $f \in A^p$ then*

$$\lim_{m\to\infty} \frac{\int |z^m f(z)|^p \, d\nu(z)}{\int |z^m f(z)|^p \, dA(z)} = 0$$

Proof. Select any $\epsilon > 0$ and choose $0 < r < 1$ so near to 1 that

$$\int_{A_r} |g(z)|^p \, d\nu(z) \le \frac{\epsilon}{2} \int |g(z)|^p \, dA(z)$$

for all $g \in A^p$. This holds, in particular, for all g of the form $z^m f(z)$ and r is independent of m. It remains to show that for sufficiently large m

$$\int_{D_r(0)} |z^m f(z)|^p \, d\nu(z) \le (\epsilon/2) \int |z^m f(z)|^p \, dA(z).$$

Note that if $E = D_r(0) = \{z : |z| < r\}$, then $E_\eta = D_s(0) = \{|z| < s\}$ with $r < s = (r+\eta)/(1+r\eta) < 1$. By Lemma 2.1

$$\begin{aligned}\int_{D_r(0)} |z^m f(z)|^p \, d\nu(z) &\le C\|\nu\|_* \int_{D_s(0)} |z^m f(z)|^p \, dA(z) \\ &\le C\|\nu\|_* s^{mp} \int_{D_s(0)} |f(z)|^p \, dA(z)\end{aligned}$$

Now the integrals $\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta$ are increasing functions of r, so their average over $[0, s]$ with respect to the measure $r\,dr$ is less than their average over $[t, 1]$ if $s < t < 1$:

$$\frac{1}{s^2} \int_{D_s(0)} |f(z)|^p \, dA(z) \le \frac{1}{1-t^2} \int_{A_t} |f(z)|^p \, dA(z).$$

Fix any such t to obtain

$$\int_{D_r(0)} |z^m f(z)|^p \, d\nu(z) \le C\|\nu\|_* \frac{s^{mp+2}}{(1-t^2)} \int_{A_t} |f(z)|^p \, dA(z).$$

Finally, because $\int_{A_t} |f(z)|^p \, dA(z) \le t^{-mp} \int |z^m f(z)|^p \, dA(z)$ we get

$$\int_{D_r(0)} |z^m f(z)|^p \, d\nu(z) \le C\|\nu\|_* \frac{s^{mp+2}}{t^{mp}(1-t^2)} \int |z^m f(z)|^p \, dA(z).$$

Now since $s^m/t^m \to 0$ as $m \to \infty$, simply choose m so large that the expression preceding the integral is less than $\epsilon/2$. □

It was already clear that a vanishing Carleson measure cannot satisfy (2.3): it doesn't satisfy the necessary condition (2.4). But Lemma 2.2 shows something more that we need later: that for any fixed $f \in A^p$ it cannot satisfy (2.3) on any sequence of functions of the form $z^m f$. The proof actually shows that the convergence of the limit to zero is uniform in f, but we will not need this fact.

Lemma 2.3. *Let $\{z_k\}$ be a sequence of points such that the disks $D_\eta(z_k)$ are disjoint, and assume that $f \in A^p$ vanishes at all the points z_k. Let μ be a Carleson measure for A^p which vanishes off the union $\bigcup_k D_\delta(z_k)$ for some $\delta < \eta/2$. Then there is a constant C depending only on η and p such that*

$$\int |f|^p \, d\mu \le C\delta^p \|\mu\|_* \int |f|^p \, dA.$$

Proof. The main idea appears in [3], but we will sketch a proof again here. First obtain the inequality

$$\sup_{|z|<\eta/2} |f(z)|^p \leq \frac{4}{\pi\eta^2} \int_{|\zeta|<\eta} |f(\zeta)|^p \, dA(\zeta)$$

This requires only the subharmonicity of $|f|^p$. If $f(0) = 0$, apply this to $f(z)/z$ to get

$$\begin{aligned} |f(z)|^p &\leq \frac{4|z|^p}{\pi\eta^2} \int_{|z|<\eta} \left|\frac{f(\zeta)}{\zeta}\right|^p \, dA(\zeta). \\ &\leq \frac{16|z|^p}{3\pi\eta^2} \int_{\eta/2<|z|<\eta} \left|\frac{f(\zeta)}{\zeta}\right|^p \, dA(\zeta). \\ &\leq \frac{16|z|^p}{3\pi\eta^2} \frac{2^p}{\eta^p} \int_{|z|<\eta} |f(\zeta)|^p \, dA(\zeta). \end{aligned}$$

The second inequality is again because the mean over $\{\eta/2 < |z| < \eta\}$ exceeds the mean over $\{|z| < \eta\}$. Then, for $|z| < \delta < \eta/2$ we get

$$|f(z)|^p \leq C\left(\frac{2\delta}{\eta}\right)^p \frac{1}{\pi\eta^2} \int_{|z|<\eta} |f(\zeta)|^p \, dA(\zeta).$$

with $C = 16/3$. Let $\sigma_k(z) = (z_k - z)/(1 - \bar{z}_k z)$ and apply the above inequality to $f \circ \sigma_k$, where f satisfies $f(z_k) = 0$ to get

$$|f(\sigma_k(z))|^p \leq C\left(\frac{2\delta}{\eta}\right)^p \frac{1}{\pi\eta^2} \int_{D_\eta(z_k)} |f(w)|^p |\sigma_k'(w)|^2 \, dA(w).$$

We now use the fact that for $w \in D_\eta(z_k)$ we have $|\sigma_k'(w)| \leq (1+|z_k|\eta)^2/(1-|z_k|^2)$. Moreover, $|D_\eta(z_k)| = \pi\eta^2(1-|z_k|^2)^2/(1-|z_k|^2\eta^2)^2$, so we get

$$|f(\sigma_k(z))|^p \leq C\left(\frac{2\delta}{\eta}\right)^p \frac{1}{|D_\eta(z_k)|} \left(\frac{1+|z_k|\eta}{1-|z_k|\eta}\right)^2 \int_{D_\eta(z_k)} |f(w)|^p \, dA(w), \quad |z| < \delta.$$

Now, $\sigma_k(D_\delta(0)) = D_\delta(z_k)$ and so

$$|f(z)|^p \leq C\delta^p \frac{1}{|D_\eta(z_k)|} \int_{D_\eta(z_k)} |f(w)|^p \, dA(w), \quad z \in D_\delta(z_k),$$

where $C = 16[2^p(1+\eta)^2]/[3\eta^p(1-\eta)^2]$ and depends only on η and p. Integrate the last inequality with respect to μ on $D_\delta(z_k)$ and then sum on k to get the inequality in the statement of the lemma. □

We can use this lemma now to get a result like Lemma 2.2, but for a measure μ which need not be a vanishing Carleson measure.

Lemma 2.4. *Let $\{z_k\}$ be a sequence of points such that the disks $D_\eta(z_k)$ are disjoint, and assume that $f \in A^p$ vanishes at all the points z_k. Let μ be a Carleson measure for A^p which vanishes off the union $\bigcup_k D_{\delta_k}(z_k)$ for some sequence of pseudohyperbolic radii δ_k tending to 0. Then*

$$\lim_{m\to\infty} \frac{\int |z^m f(z)|^p \, d\mu(z)}{\int |z^m f(z)|^p \, dA(z)} = 0$$

Proof. Fix $0 < R < 1$ and let $\mu = \mu_1 + \mu_2$ where μ_1 is concentrated on $|z| < R$ and μ_2 is concentrated on $R \le |z| < 1$. Then μ_2 satisfies the hypothesis of Lemma 2.3 for some $\delta = \delta(R)$ that tends to 0 as R tends to 1. Thus

$$\frac{\int |z^m f|^p \, d\mu_2}{\int |z^m f|^p \, dA} \le C\|\mu\|_* \delta^p$$

and the right side can be made arbitrarily small by choosing R near to 1. Moreover, the argument in Lemma 2.2 can be used without change to show that

$$\lim_{m\to\infty} \frac{\int |z^m f|^p \, d\mu_1}{\int |z^m f|^p \, dA} = 0.$$

Combining these two observations yields the lemma. □

It might be noted that a vanishing Carleson measure must fail to be 'bounded below' on any subspace of A^p invariant under multiplication by z. Moreover, the measure μ in Lemma 2.4 must fail to be bounded below on the special invariant subspace of functions vanishing on the sequence $\{z_k\}$. This paper will not make any use of these observations.

3. The Construction of φ

The idea of the construction is to create first the 'counting function' n_φ so that the measure $n_\varphi \, dA$ is the sum of two measures satisfying the hypotheses of Lemmas 2.2 and 2.4. We therefore begin with an (η, R)-lattice that is a zero set for A^2. There are several ways to create such a lattice, but a very concrete example can be created by applying a theorem from the paper [5].

Proposition 3.1. *Let* $\{z_{nk} : n = 1,2,3\dots,\ k = 1,2,\dots 8^n\}$ *satisfy* $1 - |z_{nk}| = 8^{-n}$, *the points* $z_{n1}, z_{n2}, \dots, z_{n8^n}$ *being equally spaced on that circle. Then* $\{z_{nk}\}$ *is a zero sequence for* A^2.

Proof. This is just Theorem 7 from [5] with $\beta = 8$, $\gamma = 1$ and $p = 2$, the condition there being that $\gamma/\log\beta < 1/p$. This is clear because $\log 8 > 2$. □

Straightforward estimates show that the circles with radii $1 - 8^{-n}$ are separated by a pseudohyperbolic distance at least 7/9 but less than 7/8. Moreover, *very* tedious estimates show that the points of $\{z_{nk}\}$ on a single such circle are separated by a pseudohyperbolic distance at least 0.89, but less than 0.96 . Thus, we may crudely estimate that $\{z_{nk}\}$ is an (η, R)-lattice if $\eta < 7/18$ and $R > 367/368$. In the construction below, let such an η and R be fixed.

Let each z_{nk} thus obtained be the center of a pseudohyperbolic disk $D_{nk} = D_{\delta_n}(z_{nk})$, where $\delta_n \le \eta$, and $\delta_n \to 0$. If necessary, rotate the positions of the z_{nk} so that none of the D_{nk} intersect the positive real axis. For each n choose an integer M_n so that

$$1 < M_n \frac{|D_{nk}|}{|D_\eta(z_{nk})|} < 2. \tag{3.1}$$

The function φ we will construct will have $n_\varphi = M_n$ on D_{nk}. Now let A_n be an annulus containing the circle of radius $1 - 8^{-n}$. The annuli are chosen to be so thin that they are disjoint, and so that if $\nu = \sum_n M_n \chi_{A_n} \, dA$, then ν is a vanishing Carleson measure. For this, it is sufficient to make the thickness of A_n be $o(1/M_n)$. Our constructed φ will have $n_\varphi = M_n$ almost everywhere on A_n. Lastly, let G be the domain in $\mathbf{D}$ bounded below by the x-axis and above by the curve $y = x(1-x)^2$. Then $\chi_G \, dA$ is a vanishing Carleson measure. Our φ will satisfy $n_\varphi = 1$ on $G \setminus \left(\bigcup_n A_n \cup \bigcup_{n,k} D_{nk} \right)$.

It is clear, if we succeed in constructing such a φ, that the measure $n_\varphi \, dA$ will satisfy condition (2.4), because any pseudohyperbolic disk of radius exceeding $(R+\eta)/(1+R\eta)$ will contain one of the disks D_{nk}. However, the restriction of the measure $n_\varphi \, dA$ to the union of those disks will satisfy the condition of Lemma 2.4, while the restriction to the union of the annuli A_n and G will satisfy the condition of Lemma 2.2. Thus it will not satisfy condition (2.3). Because of (3.1) it will satisfy condition (2.2). Thus, the function φ will satisfy conditions (1.1) and (1.4), but the composition operator C_φ will not have closed range.

Let $W \stackrel{\text{def}}{=} \bigcup_{n,k} D_{nk} \cup \bigcup_n A_n \cup G$ and consider it's inverse image under the exponential function. Naturally, the exponential function will have an infinite number of points that map to any given point in W. Therefore we discard enough of W so that the number of preimages matches our desired n_φ. Discard all but one copy of the preimage of G, keeping only the one that stands just above the negative real axis; discard all but M_n copies of the preimage of each D_{nk}, keeping only those between the x-axis and the line $\operatorname{Im} z = 2M_n\pi$. Finally, the preimage of each annulus A_n is an infinite vertical strip; we keep only the points in each such strip with imaginary part between 0 and $2M_n\pi$. Let V denote the set of points not discarded.

If we think of the preimage of G as a sloping hillside, then V is a set consisting of that hillside, plus an infinite number of tall, thin stalks (the preimages of the A_n) growing up along the vertical lines $\operatorname{Re} z = \log(1-8^{-n})$ to height $2M_n\pi$ and bearing a large number ($8^n M_n$) of equally spaced fruits (the preimages of the disks D_{nk}).

The set V is a simply connected domain. If ψ is the conformal map from $\mathbf{D}$ onto this domain, then $\varphi = \exp \circ \psi$ will have the same number of preimages in $\mathbf{D}$ as the exponential function on V, so φ is the desired mapping.

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