C^2 Extensions of Analytic Functions Defined in the Complex Plane

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Introduction. Analytic continuations to C^n of solutions to elliptic differential equations defined over domains in R^n have been studied in a general context by Kiselman [5].For the case where the elliptic operator is the Laplacian a detailed analysis of these analytic continuations have been given by Lelong [6], Siciak [10] and Jarnicki [4]. Related results have also been described by Hayman [2]. In this paper we consider some of the properties of analytic continuations to C^2 of analytic functions defined over domains in the complex plane.The main result proved is:

Theorem 1. *Suppose that Ω is a domain lying in the complex plane.Then there exist two unbounded domains Ω_1 and Ω_2 lying in $C^2$and containing Ω, such that for each analytic function $f(z)$ defined on Ω there are two holomorphic functions*

$$f_j : \Omega_j \to C^2 \qquad j = 1, 2,$$

which take their values in two complex subplanes of C^2, and such that on the domain Ω

$$f = f_1 + f_2,$$

and f_j is not identically zero for either f_1 or f_2, unless

$$f(z) = 0.\blacksquare$$

Our approach hinges on considering the complex numbers as a two dimensional algebra over the real numbers, and complexifying this algebra to obtain a two dimensional algebra over the complex numbers. This enables us to extend the classical Cauchy integral formula to an integral formula in C^2, and to factorize the Cauchy kernal into two separate kernals. Other related results on domains in C^2 are described in the book of G. Baily Price [8] and the work of Rochon [9].

Notation: We shall denote the set of matrices

$$\left\{ \begin{pmatrix} a_1 a_2 \\ a_3 a_4 \end{pmatrix} : a_j \in R \textit{ where } j = 1, 2, 3, 4 \right\}$$

by $R(2)$. Under matrix multiplication and addition the set $R(2)$ is an algebra over the field of real numbers. We shall denote the set of matrices

$$\left\{ \begin{pmatrix} b_1 b_2 \\ b_3 b_4 \end{pmatrix} : b_j \in C \textit{where } j = 1, 2, 3, 4 \right\}$$

by $C(2)$. Under matrix multiplication and addition the set $C(2)$ is an algebra over the field of complex numbers.
It may be observed that $R(2)$ is a real subalgebra of $C(2)$.For the matrix

$$E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in R(2) \text{ we have that } E^2 = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Consequently, the complex field may be identified canonically with the real, two dimensional subalgebra,

$$R\begin{pmatrix}1&0\\0&1\end{pmatrix}+R\begin{pmatrix}0&-1\\1&0\end{pmatrix}, \text{ of } R(2).$$

Henceforth, we shall denote $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ by I. The real algebra $RI+RE$ is a real subalgebra of the complex, two dimensional algebra

$$CI+CE=\{z_1I+z_2E : z_1, z_2 \in C\} \subseteq C(2)$$

On considering the matrices $E_1 = \frac{1}{2}(I+iE)$ and $E_2 = \frac{1}{2}(I-iE)$ we have

$$E_1E_2 = E_2E_1 = 0,$$

$$E_1^2 = E_1, E_2^2 = E_2$$

On placing $\acute{z}_1 = z_1 - iz_2$ and $\acute{z}_2 = z_1 + iz_2$ it may be observed that

$$z_1I + z_2E = \acute{z}_1E_1 + \acute{z}_2E_2.$$

Consequently

$$CI+CE = CE_1 + CE_2.$$

As demonstrated in expression (1) the matrices E_1 and E_2 are mutually annihilating idempotents of the algebra CE_1+CE_2. Consequently the complex algebra $CI+CE$ is isomorphic to the algebra $C+C$. As shown in [1, part 1] the complex algebra $C+C$ is the first example of a complex Clifford algebra.

From the above remarks we may for each point $x+iy \in C$ consider the point $xI+yE \in CI+CE$, and for each analytic function $f(z) = u(x,y)+iv(x,y)$ define on a domain $\Omega \subseteq C$ we have the Cauchy integral formula.

$$f'(x_0I+y_0E) =$$

$$\frac{E^{-1}}{2\pi}\int_\Gamma \frac{((x-x_0)I-(y-y_0)E)}{(x-x_0)^2+(y-y_0)^2}(u(x,y)I+v(x,y)E)(dxI+dyE),$$

where $f'(x_0I+y_0E) = u(x_0,y_0)I+v(x_0,y_0)E, x_0+iy_0 \in \Omega$, and Γ is a one cycle in $[(RI+RE)/\{x_0I+y_0E\}] \cap \Omega'$, where $\Omega' = \{x'I+y'E : x+iy \in \Omega\}$,and is homologous within $\Omega'/\{x_0I+y_0E\}$ to a circle centered at x_0I+y_0E.

Notation: For each point $(z_1I+z_2E) \in CI+CE$ we donate the null cone

$$\left\{(z_3I+z_4E)\,CI+CE : (z_1-z_3)^2+(z_2-z_4)^2=0\right\}$$

by $N(z_1I+z_2E)$. It may be observed that

$$N(z_1I+z_2E) = N^+(z_1I+z_2E) \cup N^-(z_1I+z_2E)$$

where

$$N^{+}(z_1 I + z_2 E) = \{(z_1 + z) I + (z_2 + iz) E : z \in C\}$$

and

$$N^{-}(z_1 I + z_2 E) = \{(z_1 + z) I + (z_2 - iz) E : z \in C\}$$

Using the integral formula (2) we may now deduce the following extension theorem:

Theorem 2. *Suppose Ω' is a domain lying in $RI + RE$ and $f' : \Omega' \to RI + RE$: $f'(x, y) = u(x, y) I + v(x, y) E$, where $u(x, y) + iv(x, y)$is an analytic function. Then on the domain Ω', where Ω' is the component of $(CI + CE) \setminus X(\Omega')$ containing Ω' , and*

$$X(\Omega') = \bigcup_{(xI+yE)\in\Omega'} N(xI + YE),$$

there is an analytic function

$$f^{\sim\prime} : \Omega' \to CI + CE$$

such that

$$\tilde{f}' \mid \Omega' = f'.$$

Proof: Suppose that $\{M_j\}_{j=1}^{\infty}$ is a sequence of compact connected two dimensional manifolds contained the the domain Ω', such that $\cup_{j=1}^{\infty} M_j = \Omega'$ and $M_j \subseteq \overset{\circ}{M}_{j+1}$. From equation (2) we have that for each point $x_0 I + y_0 E \in M_j$

$$f'(x_0, y_0) = \frac{E^{-1}}{2\pi} \int_{\partial M_j} \frac{((x - x_0) I - (y - y_0) E)}{(x - x_0)^2 + (y - y_0)^2} f'(x_0, y_0)(dxI + dyE).$$

Suppose now that $U(M_j)$ is the component of $(CI + CE) \setminus X\left(\overset{\circ}{M}_j\right)$ containing $\overset{\circ}{M}_j$, then the integral

$$\frac{E^{-1}}{2\pi} \int_{\partial M_j} \frac{((x - z_0) I - (y - w_0) E)}{(x - z_0)^2 + (y - w_0)^2} f'(x, y)(dxI + dyE)$$

where $(z_0 I + w_0 E) \in U(M_j)$, determines a function, f'_j, on $U(M_j)$which is analytic in two complex variables. Moreover,

$$\tilde{f}_j \mid_{\overset{\circ}{M}_j} = f'$$

By the arguments given in [6] it may be observed that for each point $z_0 I + w_0 E \in U(M_j) \setminus \overset{\circ}{M}_j$, the set

$$N(z_0 I + w_0 E)(RI + RE)$$

consists of precisely two points. As M_j is a connected manifold we may deduce from taking a path

$$h : [0, 1] \rightarrow U(M_j),$$

where $h(0) = (x_0 I + y_0) \in \overset{\circ}{M}_j$ and $h(1) = z_0 I + w_0 E$, that

$$N(z_0 I + w_0 E) \cap (RI + RE) \subseteq \overset{\circ}{M}_j.$$

As $M_j \subseteq \overset{\circ}{M}_{j+1}$ it now follows that

$$\overset{\sim\prime}{f}_{j+1|U(M_j)} = \overset{\sim\prime}{f}_j.$$

On placing $\tilde{f}_j = \tilde{f}$, for each j, we obtain the result.

Observation: On placing $\overset{\sim\prime}{f}(z_0, w_0) = \tilde{u}(z_0, w_0) I \tilde{v}(z_0, w_0) E$. we may observe that this function satisfies the generalized Cauchy-Riemann equations:

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial z_o} &= \frac{\partial \tilde{u}}{\partial w_0} \\ \frac{\partial \tilde{u}}{\partial w_0} &= \frac{\partial \tilde{u}}{\partial z_0} \end{aligned}$$

In the case where Ω' is a disc of radius r, and centered at the origin, the domain Ω' is the Lie ball described in [7], this domain is Cartan's classical domain type four (see[3]).

We may now use theorem 2 to prove theorem 1.

Proof of Theorem 1: We first observe that the integral formula (3) factorizes into

$$\frac{E^{-1}}{2\pi} \int_{\partial m_j} \left(\frac{E_1}{(x - z_0) - i(y - w_0)} + \frac{E_2}{(x - z_0) + i(y - w_0)} \right) f'(x, y)(dxI + dyE)$$

Consequently, we have that on the domain $U(M_j)$ the function $\overset{\sim\prime}{f}$ is equal to $\tilde{f}_{1,j} + f_{2,j}$. where

$$f_{1,j}(z_0, w_0) = \frac{E^{-1}}{2\pi} \int_{\partial M_j} \frac{E_2}{(x - x_0) - i(y - w_0)} f'(x, y)(dxI + dyE)$$

and

$$f_{2,j}(z_0, w_0) = \frac{E^{-1}}{2\pi} \int_{\partial M_j} \frac{E_2}{(x - z_0) + i(y - w_0)} f'(x, y)(dxI + dyE).$$

Moreover, the kernal $[(x - z_0) - i(y - w_0)]^{-1}$ is a non-singular provided $(x - z_0)$ is not equal to $(iy - iw_0)$, and the kernal $[(x - z_0) + i(y - w_0)]^{-1}$ is non-singular provided $(x - z_0)$ is not equal to $(iw_0 - iy)$.
Consequently, the holomorphic function

$$\tilde{f}_{1,j}(z_0, w_0) = \frac{E^{-1}}{2\pi} \int_{\partial M_j} \frac{E_1}{(x - x_0) - i(y - w_0)} f^{\sim}(x, y)(dxI + dyE)$$

is well defined on the domain $U^+\left(\overset{\circ}{M}_j\right)$, where $U^+\left(\overset{\circ}{M}_j\right)$ is the component of $(CI + CE)\backslash X^+\left(\overset{\circ}{M}_j\right)$ containing $\overset{\circ}{M}_j$, and

$$X^+\left(\overset{\circ}{M}_j\right) = \frac{U}{(xI + yE) \in \partial M_j} N^+(xI = yE).$$

Also, the holomorphic function

$$\tilde{f}_{2,j}(z_0, w_0) = \frac{E^{-1}}{2\pi} \int_{\partial M_j} \frac{E_2}{(x - z_0) - I(y - w_0)} f'(x, y)(dxI + dyE)$$

is well defined on the domain $U^-\left(\overset{\circ}{M}_j\right)$, where $U^-\left(\overset{\circ}{M}_j\right)$ is the component of $(CI + CE)\backslash X^-\left(\overset{\circ}{M}_j\right)$ containing $\overset{\circ}{M}_j$, and

$$X^-\left(\overset{\circ}{M}_j\right) = \frac{U}{(xI + yE) \in \partial M_j} N^-(xI = yE).$$

It is straightforward to observe that the functions $\tilde{f}_{1,j}$and $\tilde{f}_{2,j}$ are continuations of the functions $f_{1,j}$ and $f_{2,j}$ respectively, to the domains $U + \left(\overset{\circ}{M}_j\right)$ and $U^-\left(\overset{\circ}{M}_j\right)$, and that these two domains are unbounded.

It is also straightforward to observe the arguments given in [6] that for each point $z_1I + z_2E \in CI + CE$ that we have the set

$$(RI + RE) \cap N^+(z_1I + z_2E)$$

contains exactly one point.

By similar arguments to those given in the proof of theorem 2 it may now be deduced that $U^+\left(\overset{\circ}{M}_j\right) \subseteq U^+\left(\overset{\circ}{M}_{j+1}\right)$ and $U^-\left(\overset{\circ}{M}_j\right) \subseteq U^-\left(\overset{\circ}{M}_{j+1}\right)$. It follows from Stokes' theorem that

$$\begin{aligned} & \frac{E^{-1}}{2\pi} \int_{\partial M_{j+1}} \frac{E_1}{(x - z_0) - i(y - w_0)} f'(x, y)(dxI + dyE) \\ = & \frac{E^{-1}}{2\pi} \int_{\partial M_j} \frac{E_1}{(x - z_0) - i(y - w_0)} f'(x, y)(dxI + dyE) \end{aligned}$$

where $z_0 I + w_0 E \in U^+ \left(\overset{\circ}{M}_j\right)$, and

$$\begin{aligned} & \frac{E^{-1}}{2\pi} \int_{\partial M_{j+1}} \frac{E_2}{(x - z_0') + (y - w_0')} f'(x, y)\,(dxI + dyE) \\ = \; & \frac{E^{-1}}{2\pi} \int_{\partial M_j} \frac{E_2}{(x - z_0') + i\,(y - w_0')} f'(x, y)\,(dxI + dyE) \end{aligned}$$

where $z_0' I + w_0' E U^- \left(\overset{\circ}{M}_j\right)$.

On placing $\Omega_1 = \bigcup_{j=1}^{\infty} U^+ \left(\overset{\circ}{M}_j\right)$ and $\Omega_2 = \bigcup_{j=1}^{\infty} U^- \left(\overset{\circ}{M}_j\right)$ it follows from expressions (4) and (5) that there is a holomorphic function $f_1 : \Omega_1 \to (CI + CE)\, E_1$, and a holomorphic function $f_2 : \Omega_2 \to (CI + CE)\, E_2$ such that

1. $$f_1 \,\big|_{u+\left(\overset{\circ}{M}_j\right)} = \widetilde{f}_{1,j}$$

2. $$f_2 \,\big|_{u-\left(\overset{\circ}{M}_j\right)} = \widetilde{f}_{2,j}$$

3. on $\Omega' \subseteq \Omega_1 \cap \Omega_2$ we have

$$f_1 + f_2 = f'.$$

It also follows directly from expressions (4) and (5) that if either f_1 or f_2 is identically zero then f' is identically zero. Consequently, we obtain the result. ■

Observation: It may be observed from the two different sets of singularities of the kernels $[(x - z_0) - i\,(y - w_0)]^{-1}$ and $[(x - z_0) + i\,(y - w_0)]^{-1}$ that the domains Ω_1 and Ω_2 are not the same in general.

Example: Suppose that $f'(xI + yE) = \left[(x - 1)^{-1} + yE\right]^{-1}$, then it may be deduced that

$$f_1 (zI + wE) = \frac{E_1}{(z - 1) + iw}$$

for $iw \neq 1 - z$, and

$$f_2 (zI + wE) = \frac{E_2}{(z - 1) + iw}$$

for $iw \neq z - 1$.

Observation: By considering the linear map $L : CI + CE \to C$ defined by $L(I) = 1$ and $L(E) = i$ we have that $L(E_1) = 0$ and $L(E_2) = 1$. Consequently $L(f_1) = 0$ and the function $L(f_2)$ is the holomorphic function

$$\widetilde{f} : L(U_2) \to C,$$

where $\widetilde{f}(z_1, z_2) = f(z_1 + iz_2)$ and $z_1 + iz_2 \in \Omega$.

References.

[1] M.F. Atiyah, R. Bott and A. Shapiro 'Clifford modules' Topology 3, 1965, 3-38.

[2] W.I. Hayman 'Power Series for Expansions for Harmonic Functions' Bulletin of the London Mathematical Society 2, 1970, 152-158.

[3] L.K. Hua 'Harmonic Analysis of Functions of Several Complex Variables in Classical Domains' Translations of Math. Monographs, 6, AMS, Providence, Rhode Island, 1963.

[4] M. Jarnicki 'Analytic Continuation of Harmonic Functions' Prace Matematiczne 17, 1975, 93-104.

[5] C.O. Kiselman 'Prolongement des solutions d'une equation aux derives partilles a coefficients constants' Bull.. Soc. Math. France 97, 1969, 329-356.

[6] P. Lelong 'Prolongement analytique et singularities complexes des fonctions harmoniques' Bull. Soc. Math. Belg. 2, 1954, 10-23.

[7] M. Morimoto 'Analytic functionals on the Lie sphere' Tokyo Journal of Mathematics, 3, 1980, 1-35.

[8] G. Baily Price 'An Introduction to Multicomplex Spaces and Functions' Marcel Dekker, New York, 1991.

[9] D. Rochon 'A Bloch constant for hyperholomorphic functions' to appear.

[10] J. Siciak 'Holomorphic continuation of harmonic functions' Ann Polon Math 29, 1974, 67-73.