CAUCHY KERNELS FOR SOME CONFORMALLY FLAT MANIFOLDS

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Abstract

Here we will consider examples of conformally flat manifolds that are conformally equivalent to open subsets of the sphere S^n. For such manifolds we shall introduce a Cauchy kernel and Cauchy integral formula for sections taking values in a spinor bundle and annihilated by a Dirac operator, or generalized Cauchy-Riemann operator. Basic properties of this kernel are examined, in particular we examine links to singular integral operators and Hardy spaces.

Introduction

In a number of papers the author and his collaborators, [6, 7, 8, 9, 14, 15] use conformal transformations to develop aspects of Clifford analysis, function theory, potential theory and classical harmonic analysis over certain examples of conformally flat manifolds. As pointed out in [16] conformally flat manifolds are ones that have an atlas whose transition functions are Möbius transformations. Using Cayley transformations from R^n into S^n one may see that the sphere S^n is a conformally flat manifold. Also by factoring out a simply connected subdomain of either R^n or S^n by a Kleinian group that acts on the domain discontinuously one may construct other examples of conformally flat manifolds. These examples include cylinders, tori, real projective spaces and $S^1 \times S^{n-1}$. Cauchy kernels for Dirac operators defined for spinor bundles over all these examples of manifolds have been constructed in [6, 7, 8, 9, 14, 15]. See also [18] for the special case of the sphere.

While Cauchy kernels and Cauchy Integral Formulas for Dirac operators defined for general Riemannian manifolds have been introduced in several works including [3, 4, 11], in the context of conformally flat manifolds one may in the examples so far considered obtain very explicit formulas for the Cauchy kernels and also Green's kernels.

In this paper we turn to look at another class of examples of conformally flat manifolds where again we can obtain explicit formulas for the Cauchy kernel and other related kernels. These manifolds are obtained using Möbius transformations to glue together n-spheres. If we glue together a total of k such spheres to an n-sphere we obtain a manifold that can be seen as a sphere with k bumps, or warts, on it. In turn we might also put a finite number of bumps or warts on each wart via the same process. We can continue to place finite numbers of warts on already existing warts. Again this is done via the same gluing process. If we do not glue any of the warts to more than one sphere or wart we obtain a manifold that is homeomorphic to S^n. As such this type of manifold is simply connected. However its conformal structure is more complicated than that of the sphere. By a similar process we may also glue together several copies of R^n, or copies of spheres and euclidean spaces.

The purpose in this paper is to give an explicit construction of the Cauchy kernels for such manifolds. This construction makes use of the unique continuation property for solutions to the Dirac equation considered here. We also show that a number of known properties of the Cauchy kernel in Euclidean space readily carry over to the context considered here. This includes a Hardy space decomposition of L^p spaces for strongly Lipschitz hypersurfaces lying in the manifold. This Hardy space decomposition is expressed in terms of solutions to a Dirac equation defined on the complementary domains to the hypersurface and with appropriate continuation properties to the boundary. The techniques used here mimic existing techniques from classical harmonic analysis in Euclidean space, see for instance [17].

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Preliminaries

We will consider R^n as embedded in the real 2^n dimensional Clifford algebra Cl_n such that under Clifford algebra multiplication $x^2 = -\|x\|^2$ for each $x \in R^n$. So each non-zero vector $x \in R^n$ has a multiplicative inverse given by the Kelvin inverse of x, namely $x^{-1} = \frac{-x}{\|x\|^2}$. If $e_1, \ldots, e_n$ is an orthonormal basis for R^n then $1, e_1, \ldots, e_n, \ldots, e_{j_1} \ldots e_{j_r}, \ldots, e_1 \ldots e_n$ is a basis for Cl_n. Here $1 \leq r \leq n$ and $j_1 < \ldots j_r$. Regarding this basis as an orthonormal basis for Cl_n then the norm, $\|A\|$ of a vector $A \in Cl_n$ can be introduced in the usual way. We will also need the anti-automorphism $\sim: Cl_n \to Cl_n :\sim (e_{j_1} \dots e_{j_r}) = e_{j_r} \dots e_{j_1}$. As usual we write $\tilde{A}$ for $\sim A$.

As shown in [1] and elswhere any Möbius transformation $y = \psi(x)$ over the one point compatification of R^n can be expressed as $(ax+b)(cx+d)^{-1}$ where a, b, c and d are elements of Cl_n and satisfy certain constraints. Those constraints are detailed in [1, 7, 13] and elsewhere. If we regard R^n as embedded in R^{n+1} and then Cl_n is a subalgebra of Cl_{n+1}. Further we may regard R^{n+1} as having as orthonormal basis $e_1, \dots, e_n, e_{n+1}$. In this case we have the Cayley transformation $c(x) = (e_{n+1}x + 1)(x + e_{n+1})^{-1}$ which maps R^n homeomorphically to $S^n \backslash \{e_{n+1}\}$.

The Dirac operator in euclidean space is the differential operator $D = \Sigma_{j=1}^n e_j \frac{\partial}{\partial x_j}$. Any Cl_n valued, differentiable function f defined on a domain U in R^n is called a left Clifford holomorphic function or left monogenic function if $Df(x) = 0$ on U. If g is also defined on U and takes values in Cl_n then if g is differentiable then g is called a right Clifford holomorphic function, or right monogenic function if $g(x)D = 0$. Here $g(x)D = \Sigma_{j=1}^n \frac{\partial g(x)}{\partial x_j} e_j$. Basic properties of left and right monogenic functions are described in [2] and elsewhere. The function $G(x) = \frac{x}{\|x\|^n}$ is an example of a function that is both left and right Clifford holomorphic. It is shown in [12] and elsewhere that if $f(y)$ is left Clifford holomorphic in the variable $y = (ax+b)(cx+d)^{-1}$ then $J(\psi, x)f(\psi(x))$ is left Clifford holomorphic in the variable x where $J(\psi, x) = \frac{\widetilde{cx+d}}{\|cx+d\|^n}$. Moreover, [12], $G(\psi(x) - \psi(y)) = \tilde{J}(\psi, y)^{-1} G(x-y) J(\psi, x)^{-1}$.

In [14] the author noted that if $y = c^{-1}(x)$ where c^{-1} is the inverse of the Cayley transformation then any left Clifford holomorphic function $f(y)$ defined on a domain U in R^n is transformed to a function $J(c^{-1}, x)f(c^{-1}(x))$ defined on a domain $c^{-1}(U)$ lying on S^n. In [14] it is shown that such a function is annihilated by a Dirac operator D_s acting over S^n. An explicit expression for D_s is determined in [5], see also [8, 15]. For each $x \in c^{-1}(U)$ the expression $J(c^{-1}, x)f(c^{-1}(x))$ takes its values in a 2^n dimensional subspace of Cl_{n+1}. What we are describing is a bundle structure over $S^n \backslash \{e_{n+1}\}$. It is now time to introduce conformally flat manifolds.

Definition 1 *An n-dimensional manifold M is said to be conformally flat if there is an atlas $\mathcal{A}$ of M such that for any pair of chart maps (ϕ_1, U_1) and (ϕ_2, U_2) the transition function $\psi_{12} = \phi_2 \phi_1^{-1}$ is a Möbius transformation wherever this function is defined.*

Using Cayley transformations it is clear that S^n is an example of a conformally flat manifold.

Two n dimensional conformally flat manifolds M_1 and M_2 are said to be conformally equivalent if there is a diffeomorphism $\Psi : M_1 \to M_2$ which with respect to the atlases $\mathcal{A}_\infty$ and $\mathcal{A}_\in$ of M_1 and M_2 that is locally a Möbius transformation.

We shall now introduce appropriate spinor bundles over conformally flat manifolds. Given a conformally flat manifold M then following [7] one can identify a pair of points $(y, Y) \in (\phi_2(U_2), Cl_n)$ with either $(x, J(\psi_{12}, x)Y)$ or $(x, -J(\psi_{12}Y)$ where $y = \psi_{12}(x)$. If a choice of signs in the previous construction can be made for each pair of chart maps in $\mathcal{A}$ that is compatible then we have constructed a spinor bundle E over M and M is regarded as a conformally flat spin manifold. The sphere S^n is an example of such a manifold. In constructing such a bundle we have used the conformal weight functions that preserve Clifford holomorphy. It follows that for any subdomain U of a conformally flat spin manifold we can introduce a section $f : U \to E$ such that locally f pulls back to a left Clifford holomorphic function. Such a section is called a left Clifford holomorphic section. It also follows that we can introduce a Dirac operator D_M that locally pulls back to the euclidean Dirac operator. Moreover $D_M f = 0$ for any left Clifford holomorphic section.

For the case of the n-sphere we have a Cauchy kernel $G_s(x, y)$ for any pair of distinct points x and y on S^n. Explicitly $G_s(x, y) = \frac{x-y}{\|x-y\|^n}$. See [8, 14, 15, 18] for more details.

Construction of the Manifolds

We begin by considering two copies, A_1 and A_2 of the annulus $A(0, \frac{1}{r}, r) = \{x \in R^n : \frac{1}{r} < \|x\| < r\}$. We identify each point $x \in A_1$ with the point $-x^{-1} \in A_2$. This is done via a Möbius transformation $\psi' : A_1 \to A_2$. We now consider two copies, S_1 and S_2, of the unit sphere S^n lying in R^{n+1} Let R_1 be a copy of R^n containg the annulus A_1 and R_2 be a copy of R^n containing A_2. We have Cayley transformations $c_1 : R_1 \to S_1$ and $c_2 : R_2 \to S_2$. Let $C_1 = c_1(A_1)$ and $C_2 = c_2(A_2)$. In fact both C_1 and C_2 are annuli on the spheres S_1 and S_2 respectively. Let $\overline{B}_1$ be the copy in R_1 of the closure of the ball in R^n centered at the origin and of radius $\frac{1}{r}$. We may similarly define $\overline{B}_2$ in R_2. For $i = 1, 2$ let $S_i' = S_i \backslash c_i(\overline{B}_i)$. We adjoin S_1' and S_2' by identifying points in C_1 with points in C_2 via the Möbius transformation $\psi : C_1 \to C_2$ where $\psi = c_2 \psi' c_1^{-1}$.

In this way we have used Möbius transformations to "glue" together the two spheres S_1 and S_2. As we have only used Möbius transformations in this "gluing" process the resulting manifold M is conformally flat. We denote this manifold by $S_1 \wedge S_2(r)$. The reason for the r apearing here is that our construction of the manifold depends on our choice of r in the outer radii of the annuli A_1 and A_2. So in fact we have constructed a whole family of conformally flat manifolds.

It is a simple matter to see that the manifold $S_1 \wedge S_2(r)$ is diffeomorphic to the sphere S^n. As such it is simply connected. As mentioned in our introduction the conformally flat manifold $S_1 \wedge S_2(r)$ can be regarded as an n-sphere with a bump or wart. In fact it may fairly easily be seen using dilation and inversion that $S_1 \wedge S_2(r)$ is conformally equivalent to S^n. However, from our construction it may be seen that $S_1 \wedge S_2(r)$ is not embedded in R^{n+1}. So in this sense this class of conformally flat manifolds can be viewed as not identical to S^n. Furthermore one can vary the radii of the spheres S_1 and S_2 so that one or both are no longer the unit sphere. In this process we widen the set of conformally flat manifolds that we may consider. Furthermore we can in turn attach another sphere S_3 to either S_1 or S_2 by the same techniques that we used here to "glue" S_1 and S_2. We would denote such a manifold as $S_1 \wedge S_2 \wedge S_3(r)$. Here we will not consider attaching S_3 to both S_1 and S_2. This particular construction leads to a manifold which is a sphere with a handle.

In general we may choose a finite number of points $x_1, \ldots, x_k$ lying in the sphere S^n. We now excise k nonintersecting closed caps $C(x_j)$ from S^n. Each cap $C(x_j)$ is centered at x_j, where $j = 1, \ldots, k$. By the process just described using Cayley transforms we now may attach k spheres $S^1, \ldots, S^k$ to this open subset of S^n. Again we obtain a conformally flat manifold that is conformally equivalent to, but not identical to, the sphere S^n.

Similarly we may attach copies of R^n to S^n. To do this consider $A(0, \frac{1}{r}, \infty) = \{x \in R^n : \|x\| > \frac{1}{r}\}$. Now using a Cayley transform c we identify $A(0, \frac{1}{r}, r) \subset A(0, \frac{1}{r}, \infty)$ with the spherical annulus $c(A(0, \frac{1}{r}, r) \subset S^n \backslash c(\overline{B}(0\frac{1}{r})$. In this way we "glued" one copy of R^n to S^n. We will denote this manifold by $R_1 \wedge S_2$.

By picking finitely many points $x_1, \ldots, x_k$ on S^n we may adapt the process just outlined, and attach finitely many copies of R^n to S^n. Using Kelvin inversion and other Möbius transformations one may see that such a manifold is conformally equivalent to $S^n \backslash \{x_1, \ldots, x_k\}$.

Of course one may repeat this process and "glue" copies of either R^n

or S^n to the spheres or copies of R^n that have already been attached to S^n, and so on. If we avoid multiply gluing a sphere or euclidean space to several parts of another sphere or euclidean space we end up with a manifold that is conformally equivalent to either S^n or $S^n\backslash\{x_1,\ldots,x_k\}$. For these types of conformally flat manifolds we are able to construct a Cauchy kernel.

Construction of the Cauchy Kernel

Here we will work simply with the case $M = S_1 \wedge S_2(r)$. All the other cases are relatively straightforward extensions of this one case. We will denote the kernel by $C_M(x,y)$. This will be a Cl_{n+1} valued function defined on $M \times M\backslash\{(x,x) : x \in M\}$. When x and y both belong to S'_j, for $j = 1,2$, then $C_M(x,y)$ may be identified with $\frac{x-y}{\|x-y\|^n}$. Here we are regarding each S'_j as being embedded in a copy of R^{n+1}. Let us denote the part of S'_1 that is identified with part of S'_2 by S_{12}, and the part of S'_2 that is identified with part of S'_1 with S'_2 by S_{21}. Recall that S_{12} and S_{21} are annuli on spheres that are identified with each other via a Möbius transformation $\psi_{12} : S_{12} \to S_{21}$. Let $M_1 = S'_1\backslash S_{12}$ and $M_2 = S'_2\backslash S_{21}$. Suppose now that $x \in S'_1$ and y belongs to the part of $S_1 \wedge S_2(r)$ where S_{12} and S_{21} are identified. Then in this case y can be identified with $y_1 \in S_{12}$ or $y_2 \in S_{21}$ where $\psi_{12}(y_1) = y_2$. In this case the kernel $C_M(x,y)$ is represented by $\frac{x-y_1}{\|x-y_1\|^n}$, which in turn may be identified with $\frac{x-\psi_{12}^{-1}(y_2)}{\|x-\psi_{12}^{-1}(y_2)\|^n}J(\psi_{12}^{-1},y_2)$. Furthermore if x also belongs to S_{12} then $x_2 = \psi_{12}(x)$ and $C_M(x,y)$ can be identified with $J(\psi_{12}^{-1},x_2)\frac{\psi_{12}^{-1}(x_2)-\psi_{12}^{-1}(y_2)}{\|\psi_{12}^{-1}(x_2)-\psi_{12}^{-1}(y_2)\|^n}J(\psi_{12}^{-1},y_2) = \frac{x_2-y_2}{\|x_2-y_2\|^n}$.

Last of all we turn to the case where $x \in S'_1$ and $y \in S'_2$. In this case we note that the Möbius transformation ψ_{12} has a unique continuation to a Möbius transformation $\Psi_{12} : N_1 \to N_2$ where $N_j = S_j\backslash M_j$. In this case $y \in N_2$ and $y = \Psi_{12}(y_1)$ for some $y_1 \in N_1$. In this case $C_M(x,y) = \frac{x-\Psi_{12}^{-1}(y)}{\|x-\Psi_{12}^{-1}(y)\|^n}J(\Psi_{12}^{-1},y)$.

While we have not covered all possibilities of locations in x and y on M in this construction all remaining possibilities can be constructed easily by adapting the existing constructions presented so far here.

Concluding Remarks

Having constructed the Cauchy kernel $C_M(x,y)$ for $M = S_1 \wedge S_2(r)$ we readily have Cauchy's integral formula.

Definition 2 *A $(n-1)$-dimensional hypersurface S lying in $S_1 \wedge S_2(r)$ is called a Lipschitz hypersurface if locally S is the image under Möbius transformations of Lipschitz surfaces lying in R^n. If the Lipschitz constants for these Lipschitz surfaces lying in R^n are bounded then S is said to be strongly Lipschitz.*

Theorem 1 *Suppose that U is an open, connected subset of $M = S_1 \wedge S_2(r)$ and $f : U \to E$ is a left Clifford holomorphic section over U then for each $y \in U$ and each $(n-1)$-dimensional strongly Lipschitz hypersurface S lying in U and bounding a subdomain of U and containing y, then*

$$f(y) = \frac{1}{\omega_n} \int_S C_M(x,y) n(x) f(x) d\sigma(x)$$

where $n(x)$ is the unit vector in TM_x that is orthogonal to TS_x and point outwards from S. Here TM and TS are the tangent bundles of M and S respectively. Furthermore σ is the Lebesgue measure on S.

Having obtained this Cauchy kernel and Cauchy integral formula standard arguments developed in [10, 11] and elsewhere give us the following decomposition result.

Theorem 2 *Suppose that S is a Lipschitz hypersurface lying in $M = S_1 \wedge S_2(r)$ and the complement of S has two components, $S^{\pm}$. Then for $1 < p < \infty$ we have that*

$$L^p(S) = H^p(S^+) \oplus H^p(S^-)$$

where $L^p(S)$ is the space of E valued sections on S that are L^p integrable, and $H^p(S^{\pm})$ is the Hardy p-space of left Clifford holomorphic sections defined on $S^{\pm}$ with non-tangential extension to S belonging to $L^p(S)$.

The construction we gave in the previous section of the Cauchy kernel for the manifold $S_1 \wedge S_2(r)$ can readily be adapted to construct similar Cauchy kernels for all the types of manifolds constructed in this paper. It follows that analogues of Theorems 1 and 2 hold in those settings too.

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