Compass and straightedge
in the Poincaré disk

by

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Abstract

We describe synthetic methods for drawing in the Poincaré disk model of the hyperbolic plane.

The spirit of this article belongs to another age. Today, "geometry" is most often analytic; this is especially suited to the making of nice pictures by computer. But here we give a synthetic approach to the development of hyperbolic geometry; all our constructions are constructions with a Euclidean compass and Euclidean straightedge, and can be carried out by hand. Indeed, M.C. Escher used something like the methods we give here to produce his well known Circle Limit I, II, III and IV prints (c.f. [6]).

In [5], H.S.M. Coxeter describes a remarkable correspondence with Escher. Having met at the 1954 International Congress of Mathematics in Amsterdam, Coxeter apparently sent Escher a paper in which a drawing similar to that of figure 1 appeared. Coxeter must have been quite pleased and surprised to find a print of Circle Limit I in his mail in December 1958. It must be noted that the drawing in the paper Coxeter had sent Escher was not even as detailed as the figure below; yet Escher deduced and generalized the technique of its construction, producing an incredibly fine tesselation of the Poincaré disk.

The technique for these constructions is quite ingenious and makes use of a certain duality between circular arcs and points inside the disk and points and Euclidean lines outside the disk, described at the end of Section 1. In [5], Coxeter only incidentally describes this duality; this is not quite enough to complete the construction and here we describe a full suite of available techniques.

I believe nothing in this article can possibly be original: surely this was all well-known at the end of the nineteenth century just as it has been long forgotten at the dawn of the twenty-first. For example, W. Burnside, in his classic [2], includes many figures along the lines of figure 1 that are most likely synthetically produced; today of course, it would be most natural to make these analytically. Unfortunately, at the moment I have not found an explicit treatment of the material in this paper.[1]

The setting is the Poincaré disk model of the hyperbolic plane. We need only a few properties of this model to carry out all of our synthetic constructions:

First, the points of the model are the points in the open unit disk in the Euclidean plane. Second, geodesics are circular arcs orthogonal to the unit circle. Inversion across such an arc is to be an isometry.

The locus of points equidistant from a given point A is a Euclidean circle (though the Euclidean center of this circle is not A!). This is consistent with our other properties: this locus must be invariant under inversion through any geodesic passing through A; circles orthogonal to every geodesic through A are the only curves satisfying this requirement.

Finally, angles in the Poincaré model are Euclidean angles. This is the only consistent choice, as inversion preserves Euclidean angles.

Though this is enough to completely specify the metric (up to some normalization), we simply note that there *is* a metric satisfying these properties. Two nice,

[1] Hopefully this will be corrected by the time the article appears.

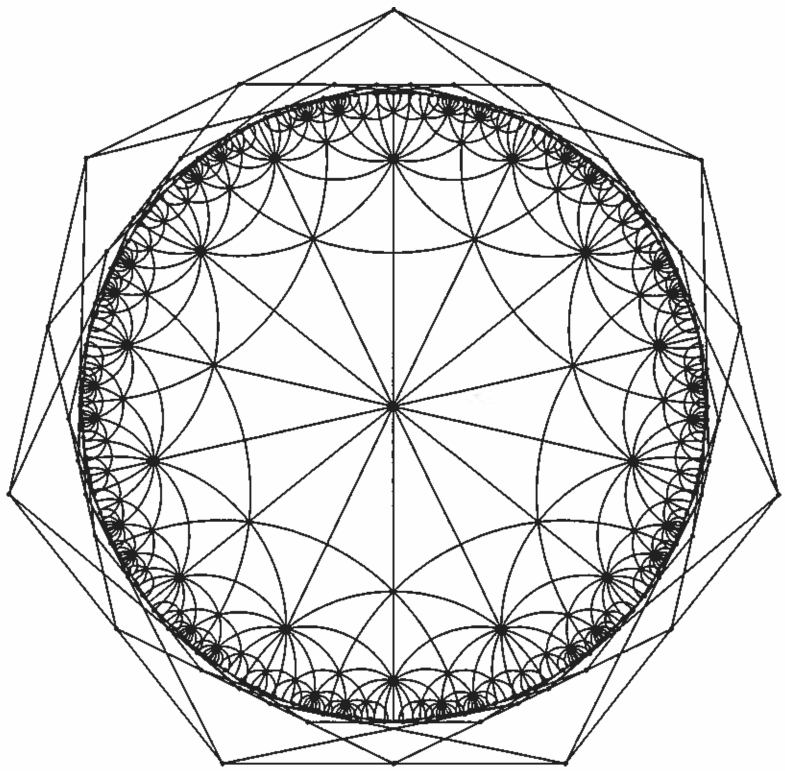

Figure 1: Synthetically constructed triangle tiling in $\mathbb{H}^2$

quite different developments of the analytic geometry from this description can be found in [4] and [9]. More analytic, but also very nice treatments can be found in [1, 3, 8] and many other sources. Interesting historical accounts of the development of non-Euclidean geometry appear in [7, 10] and elsewhere.

Some readers raised on an analytic treatment, with its abundance of hyperbolic trigonometric functions, natural logarithms and the like, may find the synthetic approach, using only Euclidean circles and lines, somewhat surprising.

But all facts about hyperbolic lines and circles are merely facts about Euclidean lines and circles (not respectively), via the Poincaré model. This is, of course, how Beltrami first showed that hyperbolic geometry was no less consistent than Euclidean geometry (though he used a different model).

For example, in the Poincaré disk, the existence of a unique hyperbolic geodesic between two given points is simply the existence of a unique circle, passing through the two given points, that is orthogonal to some given circle. It may not be obvious that such a circle exists— we will give the construction shortly— but the point is that we can use synthetic techniques quite readily.

In the first section, we give some basic useful constructions in the Euclidean plane. In the next section we show how to construct a hyperbolic straightedge and compass. Once these are in hand, we will give several handy constructions in the hyperbolic plane. In the final section we give techniques for constructing various tesselations of the hyperbolic plane with compass and straightedge.

Geometer's Sketchpad sketches and scripts to accompany this article can be found on the author's website, *http://comp.uark.edu/~cgstraus.*

1 Useful elementary constructions

We will take as given that the reader knows how to bisect a line segment, draw a parallel through a given point, draw a perpendicular through a given point and other elementary Euclidean constructions.

We will also need a few other slightly less elementary constructions, that we give for completeness.

Construction 1.1 *Construct a circle through three given non-collinear points* A, B, C

Construct segments $\overline{AB}$, $\overline{BC}$. Construct the perpendicular bisectors ℓ_1, ℓ_2 of these segments. Let O be the point of intersection of ℓ_1, ℓ_2; the desired circle has center O and passes through, say A. □

Construction 1.2 *Invert a point through a circle $\mathcal{C}$ with center* O

There are a variety of methods; here is one:

If our point B is outside the circle, construct segment $\overline{OB}$ as at left in figure 2. Construct the circle $\mathcal{C}'$ passing through O, with center at the midpoint of this segment. Let P be a point of intersection of $\mathcal{C}$ and $\mathcal{C}'$ (note that $\triangle$OPB is a right triangle). Now take the perpendicular ℓ to $\overline{OB}$ through P. Let A be the point of intersection of ℓ with $\overline{OB}$. Then A will be the inverse of B with respect to $\mathcal{C}$.

If we wish to invert a point A lying inside the circle (other than O!) simply reverse this process. Take the perpendicular ℓ through A to $\overline{OA}$. Let P be the point of intersection of ℓ and $\mathcal{C}$. Let ℓ' be the perpendicular to $\overline{OP}$ passing through P. Then the point B of intersection of $\overleftrightarrow{OA}$ and ℓ' is the desired inverse of A through $\mathcal{C}$. □

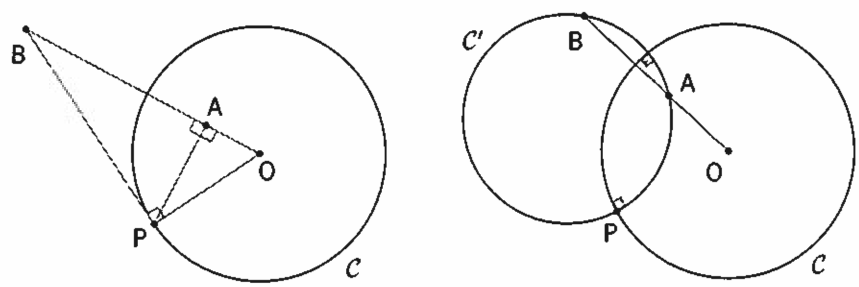

Figure 2: Constructions 1.2 and 1.3

There is a significantly quicker method if a little additional structure is given:

Construction 1.3 *Given a pair of orthogonal circles $\mathcal{C}, \mathcal{C}'$ with centers* O, O′ *respectively, and a point* A *on $\mathcal{C}'$, construct the inverse of* A *with respect to $\mathcal{C}$.*

Simply construct the ray $\overrightarrow{OA}$ as at right in figure 2; one of the two points of intersection of $\overrightarrow{OA}$ with $\mathcal{C}'$ will be A; the other, B, will be the inverse of A with respect to $\mathcal{C}$. □

One proves this gives the desired point by considering the similar triangles $\triangle$OAP, $\triangle$OPB where P is a point of intersection of $\mathcal{C}$ and $\mathcal{C}'$. It immediately follows that:

Lemma 1.4 *Let $\mathcal{C}, \mathcal{C}'$ be orthogonal circles and let* A *be a point on $\mathcal{C}'$. Then the inverse of* A *with respect to $\mathcal{C}$ also lies on $\mathcal{C}'$. Similarly, if* A *and* B *are inverses with respect to $\mathcal{C}$, lying on a circle $\mathcal{C}'$ then $\mathcal{C}$ and $\mathcal{C}'$ are orthogonal.*

Consequently:

Lemma 1.5 *Let $\mathcal{C}$ be a circle with center O. Let A be a point other than O. Then the locus of centers of all circles passing through A and orthogonal to $\mathcal{C}$ is a straight line. If A is in the interior of [on, in the exterior of] $\mathcal{C}$ this line is in the exterior of [tangent to, in the exterior of] $\mathcal{C}$. Finally, any line in the exterior of, or tangent to, $\mathcal{C}$ is such a locus.*

Proof This is simply because any circle through A and orthogonal to $\mathcal{C}$ must pass through the inverse B of A with respect to $\mathcal{C}$. Consequently, if A, B are distinct (i.e. not on $\mathcal{C}$), the center of this circle must lie on the perpendicular bisector of $\overline{\mathsf{AB}}$, which is in the exterior of $\mathcal{C}$. Conversely, any circle with center on this bisector passes through both A and B and so is orthogonal to $\mathcal{C}$.

In the special case where A lies on $\mathcal{C}$, then the locus is clearly the line tangent to $\mathcal{C}$ at A.

It is trivial to show that any line in the exterior of, or tangent to $\mathcal{C}$ is such a locus. □

We further note:

Construction 1.6 *Given a circle $\mathcal{C}$ with center O, and point A in the exterior of $\mathcal{C}$, construct the unique circle $\mathcal{C}'$ with center A, orthogonal to $\mathcal{C}$.*

Let $\mathcal{C}''$ be the circle with center at the midpoint of segment $\overline{\mathsf{AO}}$ and passing through A, O, as at left in figure 3; let P be a point in the intersection of $\mathcal{C}$ and $\mathcal{C}'$. Then the desired circle has center A and passes through P. □

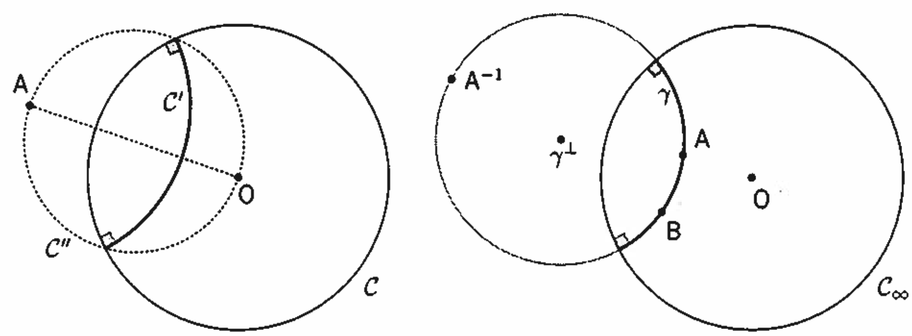

Figure 3: Constructions 1.6 and 2.1

Note then that Lemma 1.5 and Construction 1.6 establish a very important fact: with respect to a given fixed circle $\mathcal{C}$, orthogonal circles $\mathcal{C}'$ are in correspondence with points in the exterior of $\mathcal{C}$ (the centers of the $\mathcal{C}'$, called their **poles**) and points in the interior of $\mathcal{C}$ correspond to straight lines (called their **polars**) in the exterior of $\mathcal{C}$.

Assuming, then, that a specific circle $\mathcal{C}$ has been fixed (as will be the case shortly), denote the pole of a circle γ by $\gamma^{\perp}$; conversely, each point B in the exterior of $\mathcal{C}$ will be the pole of some circle, denoted $\mathsf{B}^{\perp}$, as illustrated in figure 4. Similarly, denote the polar of each point A in the interior of $\mathcal{C}$ by $\mathsf{A}^{\perp}$; conversely, each line ℓ not meeting $\mathcal{C}$ is the polar of some point, denoted $\ell^{\perp}$.

(Properly speaking, the poles of our geodesics in fact reside in the real projective plane $\mathbb{R}P^2$: we must adjoin points at infinity to account for the poles of those geodesics which are diameters of $\mathcal{C}$. As a further aside, this gives a simple proof that the space of lines in $\mathbb{H}^2$ is an open Mobius band, i.e. $\mathbb{R}P^2$ less the closed disk bounded by $\mathcal{C}$.)

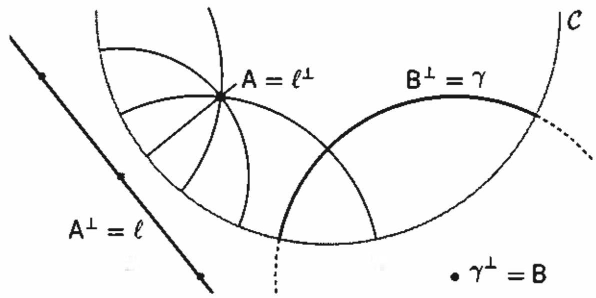

Figure 4: Poles and polars

2 Constructing the Hyperbolic Straightedge and Compass

Now we turn to the constructions of the hyperbolic straightedge and compass. We will always take as given $\mathcal{C}_\infty$, the circle at infinity in the Poincaré disk model of the hyperbolic plane and O, the center of this circle. The open disk itself will be denoted D. When we refer to inverses, unless otherwise noted, we will mean inverses with respect to $\mathcal{C}_\infty$; and $X^\perp$ will mean the polar, pole, etc. of some object X with respect to $\mathcal{C}_\infty$, in the sense described above.

Construction 2.1 *Given points* $\mathsf{A}, \mathsf{B} \in D$*, construct the hyperbolic geodesic* $\overleftrightarrow{\mathsf{AB}}$*. Equivalently, given two points* A, B *and a circle* $\mathcal{C}_\infty$ *with center* O*, construct the unique circle through* A, B *that is orthogonal to* $\mathcal{C}_\infty$*.*

This is wonderfully simple: As at right in figure 3, invert A through $\mathcal{C}_\infty$ to construct A^{-1} and the desired circle is the circle γ through A, B and A^{-1}. Note that $\gamma^\perp$ is the midpoint of $\overline{\mathsf{AA}^{-1}}$. Note that by Lemma 1.4, the constructed circle is orthogonal to $\mathcal{C}_\infty$. □

So note, then, that hyperbolic geodesics are in precise correspondence with points outside the Poincaré disk, and points within the Poincaré disk are in precise correspondence with (Euclidean) lines in the exterior of the disk. This gives us a very rapid method of constructing the geodesic between two points A, B *if the polars* $\mathsf{A}^\perp$ *and* $\mathsf{B}^\perp$ *happen to be provided*:

Construction 2.2 *Given points* $\mathsf{A}, \mathsf{B} \in D$ *and their polars* $\mathsf{A}^\perp$ *and* $\mathsf{B}^\perp$*, construct* $\overleftrightarrow{\mathsf{AB}}$.

As illustrated at left in figure 11, letting P be the point of intersection of $\mathsf{A}^\perp$ and $\mathsf{B}^\perp$, the desired geodesic is simply $\mathsf{P}^\perp$, the circle with center P passing through A, B.

In practice, if A and B are relatively close to each other in D, $\mathsf{A}^\perp$ and $\mathsf{B}^\perp$ will be nearly parallel and P will be hard to locate precisely. Fortunately, the perpendicular bisector of the (Euclidean) segment $\overline{AB}$ must also pass through P, increasing the accuracy of the construction tremendously. □

We now turn to hyperbolic circles. This is a little more involved. Before we construct the hyperbolic compass, we need the following:

Construction 2.3 *Given two orthogonal circles $\mathcal{C}_1, \mathcal{C}_2$, with centers O_1, O_2, and a point A on $\mathcal{C}_1$, construct the unique circle $\mathcal{C}$ passing through A that is orthogonal to both $\mathcal{C}_1, \mathcal{C}_2$.*

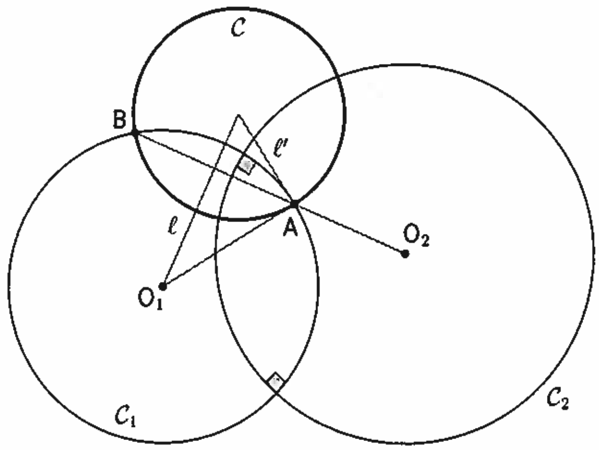

Figure 5: Construction 2.3

As in figure 2.3, let B be the other point besides A in $\mathcal{C}_1 \cap \overleftrightarrow{O_2A}$. Note, since $\mathcal{C}_1$ and $\mathcal{C}_2$ are orthogonal, that A and B are inverses of one another through $\mathcal{C}_2$. Consequently any circle through both A and B will be orthogonal to $\mathcal{C}_2$; a circle passes through A and B if and only if its center is on ℓ, the perpendicular bisector of $\overline{AB}$. Let ℓ' be the tangent to $\mathcal{C}_1$ at A. Any circle with center on ℓ' is orthogonal to $\mathcal{C}_1$. Consequently, the desired circle has center at the intersection of ℓ, ℓ' and passes through A. □

Construction 2.4 *Given points* A, B *construct the hyperbolic circle with hyperbolic center* A*, passing through* B.

We will take as given that the desired curve is the unique Euclidean circle that is orthogonal to every geodesic through A. Essentially we apply the previous construction twice (figure 6):

First construct $\overleftrightarrow{AB}$. Next construct the circle $\mathcal{C}'$ orthogonal to $\overleftrightarrow{AB}$ and $\mathcal{C}_\infty$ passing through A. Finally, the desired circle $\mathcal{C}$ is the circle orthogonal to $\overleftrightarrow{AB}$ and $\mathcal{C}'$ passing through B. □

3 Several elementary constructions in the hyperbolic plane

A large number of useful constructions make no assumption of the parallel postulate and so work in the hyperbolic plane, substituting our hyperbolic straightedge and compass for the Euclidean compass and straightedge.

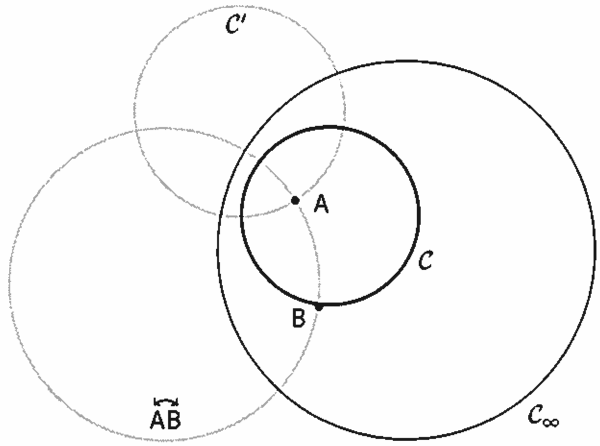

Figure 6: Construction 2.4

Construction 3.1 *Given* $A, B \in D$*, construct the perpendicular bisector of the segment* $\overset{\frown}{AB}$*. Similarly, construct the midpoint of the segment* $\overset{\frown}{AB}$*.*

The construction is precisely the same as in the Euclidean plane, but with our hyperbolic compass and straight edge. As at left in figure 7, draw the two hyperbolic circles $\mathcal{C}_1, \mathcal{C}_2$ centered at A, B and passing through B, A. Let P, Q be the points of $\mathcal{C}_1 \cap \mathcal{C}_2$. Then $\overleftrightarrow{PQ}$ is the desired perpendicular bisector and the midpoint of $\overset{\frown}{AB}$ is the intersection of $\overleftrightarrow{PQ}$ and $\overleftrightarrow{AB}$. □

In fact, though, this construction can be carried out much more efficiently:

(Alternate version) Inversion across the perpendicular bisector γ of $\overset{\frown}{AB}$ will have to interchange A and B; moreover since γ is to be orthogonal to $\mathcal{C}_\infty$, this inversion must also interchange A^{-1} and B^{-1}. Hence, $\gamma^\perp$ must lie on both $\overleftrightarrow{AB}$ and $\overleftrightarrow{A^{-1}B^{-1}}$. At right in figure 7, we take the point of intersection as $\gamma^\perp$, and consequently Construction 1.6 gives us the desired bisector γ. □

Similarly, we can easily mark off equal divisions in a line, etc. adapting the Euclidean constructions, though there may also be simpler, more direct techniques available. Here are a few more useful constructions worth noting:

Construction 3.2 *Given a point* $A \in D$ *and a geodesic* γ *construct the hyperbolic reflection of* A *across* γ*. Similarly, "translate" a given hyperbolic triangle* $\triangle ABC$ *to a given location (specified by the images* A', B' *of* A, B*).*

First, to reflect a point A across γ, simply invert A through γ, using Construction 1.3. For the second part, we need to construct the image of C; there are two possibilities, depending on whether we wish the transformation to be orientation preserving or reversing.

We follow exactly the same construction as in the Euclidean case. We first construct the perpendicular bisector γ_1 of the segment $\overset{\frown}{AA'}$. Then let B'' be the inverse of B across γ_1. Then let γ_2 be the geodesic through A' and the midpoint of $\overset{\frown}{B'B''}$. Hence inverting first through γ_1 and then through γ_2 will take A, B to A', B'. Let C' be the image of C under this operation. If we wish our transformation to be

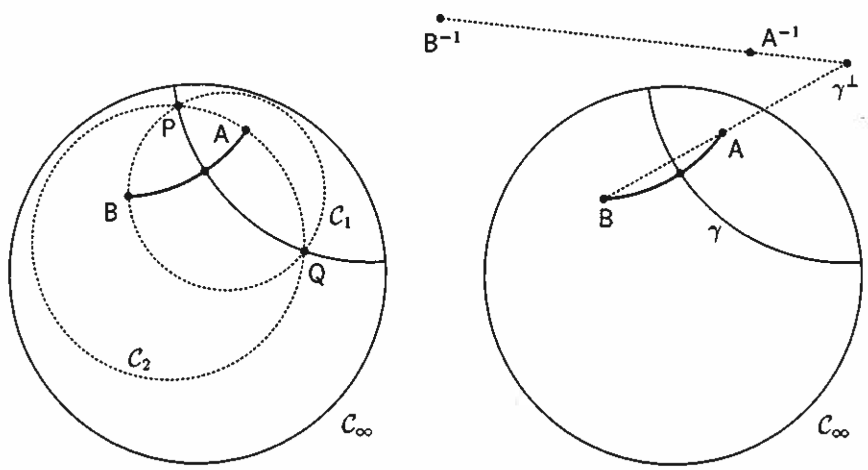

Figure 7: Construction 3.1: On the left, a modification of the usual Euclidean construction. On the right, a more efficient version.

orientation preserving we are done. Otherwise, invert once more across the geodesic $\overleftrightarrow{\mathsf{A'B'}}$. □

We next adapt the standard Euclidean proof to show that the "collapsible" hyperbolic compass of Construction 2.4 is equivalent to a "rigid" hyperbolic compass.

Construction 3.3 *Given points* $\mathsf{A}, \mathsf{B}, \mathsf{C} \in D$*, construct the hyperbolic circle with hyperbolic center* A *and radius congruent to the hyperbolic segment* $\overset{\frown}{\mathsf{BC}}$.

First construct the perpendicular bisector γ of the segment $\overset{\frown}{\mathsf{BA}}$. Then invert C across γ to obtain P. The desired circle has hyperbolic center A and passes through P. □

We conclude this section with a useful and simple construction:

Construction 3.4 *Given a geodesic* γ*, a point* $\mathsf{A} \in \gamma$*, and a Euclidean angle* α*, construct a geodesic meeting* γ *at* A *with angle* α.

As in figure 10 let ℓ be the line passing through $\gamma^{\perp}$ and A and let ℓ' be the result of rotating ℓ by α about A. Then the desired geodesic is $\mathsf{P}^{\perp}$ where P is the intersection of ℓ' and $\mathsf{A}^{\perp}$. □

4 Triangle tilings of the hyperbolic plane

We now turn to the construction of "triangle tilings" of the hyperbolic plane; there are many descriptions of such tilings and their connections. A nice early treatment (complete with synthetically drafted illustrations!) appears in [2]. Keeping things elementary and synthetic, we will simply say that such a tiling is described, up to hyperbolic isometries, by an unordered triple of whole numbers p, q, r with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. One can prove by a variety of methods that there is a tiling of the hyperbolic plane by triangles with interior angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$, generated by inversions across the sides of some base triangle.

We will give an explicit construction of such tilings, first for the triple $7, 4, 2$ and then for all triples of the form $p, q, 2$. The reader is invited to generalize.

Each case will consist of two tasks: First, construct a triangle with the appropriate angles and then, second, extend this into a tiling. For the first task, we will assume that the appropriate Euclidean angles have been provided (thus we are not asking that these angles are themselves constructible!). The second task can be carried out trivially, but quite tediously by repeated applications of Construction 3.2. However, we will give an elaboration of an elegant technique implicit in [5]. In this paper, Coxeter describes correspondence with Escher leading to the construction of Escher's famous Circle Limits I-IV. The method described in that paper, in effect repeated applicatnions of Construction 2.2, is not quite sufficient to carry on indefinitely, though, but additional tricks have been provided above. Escher must have discovered these or very similar devices on his own.

4.1 Constructing the $7, 4, 2$ triangle tiling

Our first task is to construct a hyperbolic triangle with angles $\frac{\pi}{7}, \frac{\pi}{4}, \frac{\pi}{2}$. Of course, an angle of $\frac{\pi}{7}$ is not constructible, but such an angle certainly exists! We will take this angle as given. It is convenient to construct the circle $\mathcal{C}_\infty$ as an artifact of the construction of the triangle (rather than beginning with $\mathcal{C}_\infty$).

Step 1: (figure 8) Choose two points O and A. Let ρ denote the (Euclidean) rotation about O by $\frac{2\pi}{7}$. Our first geodesic γ will be the (Euclidean) circle centered at the midpoint B of $\overline{\mathsf{A}\rho(\mathsf{A})}$ and passing through A and $\rho(\mathsf{A})$. Now we need $\mathcal{C}_\infty$: let $\mathcal{C}$ be the circle centered at the midpoint of $\overleftrightarrow{\mathsf{BO}}$ and let C be a point of $\mathcal{C} \cap \gamma$. Then let $\mathcal{C}_\infty$ be the circle with center O passing through C. We now note that:

(i) $\mathcal{C}_\infty$ and γ are orthogonal (ii) γ is orthogonal to the line $\overleftrightarrow{\mathsf{BO}}$, and (iii) γ meets the line $\overleftrightarrow{\mathsf{AO}}$ at an angle of $\frac{\pi}{7}$ (this can be seen by inspecting the angle between the lines $\overleftrightarrow{\mathsf{AO}}$ and $\overleftrightarrow{\mathsf{AB}}$).

Consequently, letting P, Q be the points of intersection of γ with $\overleftrightarrow{\mathsf{AO}}, \overleftrightarrow{\mathsf{BO}}$ in the interior of D, we have that the hyperbolic triangle $\triangle$OPQ has interior angles $\frac{\pi}{7}, \frac{\pi}{7}, \frac{\pi}{2}$.

The geodesic $\mathsf{A}^\perp$ must pass through Q (since the line $\overleftrightarrow{\mathsf{AB}}$ must be the polar of Q); it is easy to show that $\mathsf{A}^\perp$ meets $\overleftrightarrow{\mathsf{OB}}$ at an angle of $\frac{\pi}{4}$ (note, for example that $\triangle$ABP is a (Euclidean) isoceles right triangle by its construction from γ). Moreover, $\mathsf{A}^\perp$ must be orthogonal to $\overleftrightarrow{\mathsf{AO}}$. Hence, letting R be the point of intersection of $\mathsf{A}^\perp$ and $\overleftrightarrow{\mathsf{AO}}$ in the interior of D, we have that $\triangle$ORP and $\triangle$PQR have interior angles $\frac{\pi}{7}, \frac{\pi}{4}, \frac{\pi}{2}$.

Note that exactly this same procedure can be used to produce hyperbolic triangles with internal angles $\frac{\pi}{p}, \frac{\pi}{4}, \frac{\pi}{2}$ for any $p \geq 5$.

Step 2: We now need to tile the Poincaré disk by congruent images of our triangle.

The crudest, but most general (and clearly successful) method is to repeatedly invert the points O, P, Q and R across the geodesics $\overleftrightarrow{\mathsf{OR}}, \overleftrightarrow{\mathsf{OQ}}$ and $\overleftrightarrow{\mathsf{RQ}}$.

However, we can do much better. At each stage, we will automatically take advantage of the dihedral symmetry in our tiling; so we will assume that once any point, geodesic, pole or polar is created, its images under this symmetry are also created. So, we are now faced with figure 9.

Our next task is to add geodesics about the point A in figure 10. This is effectively the same as rotating the geodesic γ by $\frac{\pi k}{7}$ about A: we simply apply

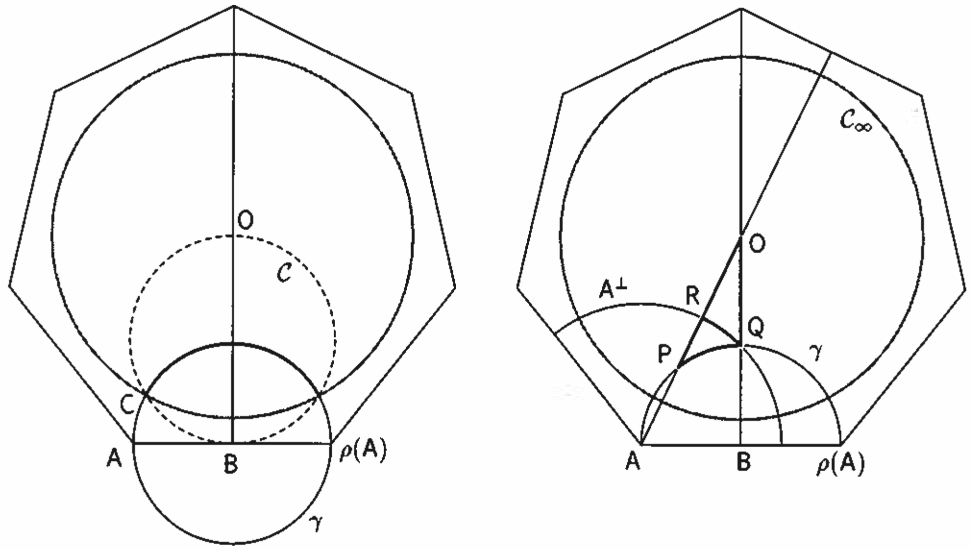

Figure 8: Constructing a 7, 4, 2 triangle

Construction 3.4 as illustrated in figure 10.

Similarly, we next need to add a geodesic between points A, B at left in figure 11. But this is just an application of our Construction 2.2. Note how much quicker this is than Construction 2.1! Alternatively, as at right in figure 11 we might go ahead and rotate the geodesic γ by $\frac{\pi k}{4}$, which gives us not only the geodesic $\overleftrightarrow{\mathsf{AB}}$ but a few others besides.

It should be clear that we can repeatedly apply these two techniques, using inversion if we ever get stuck, fleshing out both the internal and external structures, until we have generated as much of the triangle tiling as we desire. We have carried this process out a bit further in figure 1

4.2 Constructing the $p, q, 2$ triangle tiling

We only need to describe the first step, the construction of a hyperbolic triangle with internal angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{2}$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{2} < 1$.

But the technique is pretty straightforward. In figure 12 we illustrate the procedure for $p = 5, q = 8$. Begin as before with two points O, A; rotate A about O by $\frac{\pi}{p}$ to produce A′. Let P be the midpoint of $\overline{\mathsf{AA'}}$. Now rotate $\overleftrightarrow{\mathsf{AA'}}$ by $\frac{\pi}{q} - \frac{\pi}{p}$ around A to produce ℓ (note that our rotations are by signed angles), and let B be the intersection of ℓ with $\overleftrightarrow{\mathsf{OP}}$. Our first geodesic will be γ, the circle centered at B, passing through A, A′.

Let $\mathcal{C}$ be the circle passing through O, B centered at the midpoint of $\overline{\mathsf{OB}}$ and let C be a point of intersection of γ with $\mathcal{C}$. Then $\mathcal{C}_\infty$ is to be the circle centered at O passing through C.

Finally, let R be the intersection of γ with $\overleftrightarrow{\mathsf{OA}}$ and let Q be the intersection of γ with $\overleftrightarrow{\mathsf{OB}}$. It is not hard to show that the hyperbolic triangle $\measuredangle$OQR has internal angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{2}$.

Once we have this triangle, we proceed exactly as before to produce a tiling.

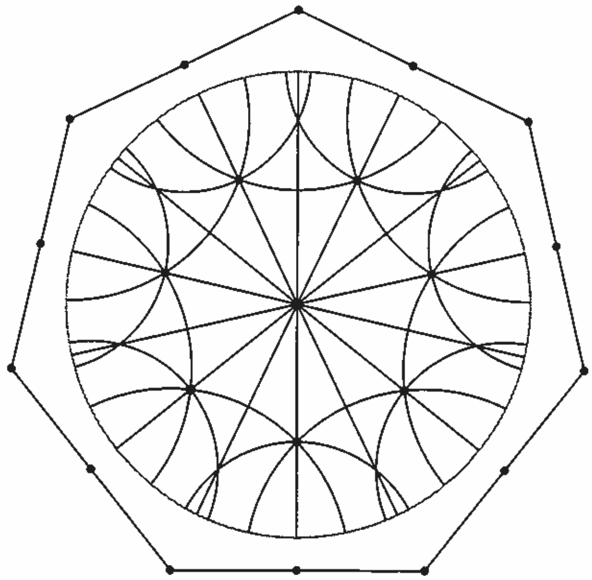

Figure 9: Beginning the 7, 4, 2 triangle tiling

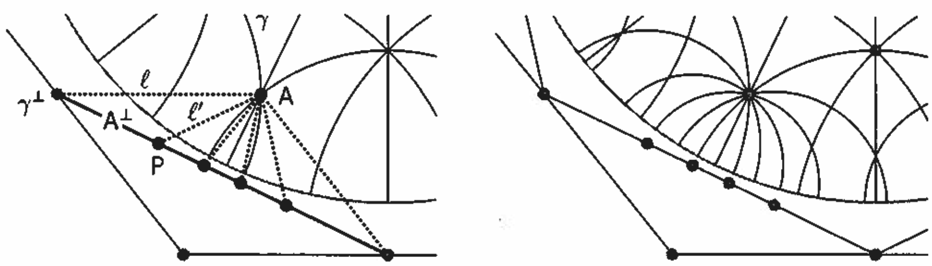

Figure 10: Rotating geodesics about a point

4.3 Variations

If we so desire, we can construct other hyperbolic triangles; Coxeter implicitly gives a method for constructing the 4, 4, 3 triangle in [5]; this method easily generalizes to give the $p, p, 3$ triangle for $p > 3$.

We can create tilings that lack the dihedral symmetry of figure 1 simply by applying Construction 3.2 to reflect our base triangle across some arbitrary geodesic before generating the tiling.

We can create tilings with more intricate shapes, based on the symmetry of the triangle tiling. Of course, this is exactly what Escher did in his Circle Limit prints.

We can easily convert our constructions to the Klein model: a geodesic in the Klein model will have the same intersection with $\mathcal{C}_\infty$ as the corresponding geodesic in the Poincaré disk. Hence, we can produce poles, polars, points, geodesics, angles, and so forth readily by converting back to the Poincaré model.

There is no end of possibilities. It is wonderfully satisfying to make these pictures by hand, patiently, with pencil and paper, compass and straightedge. I encourage you to test this theorem for yourself!

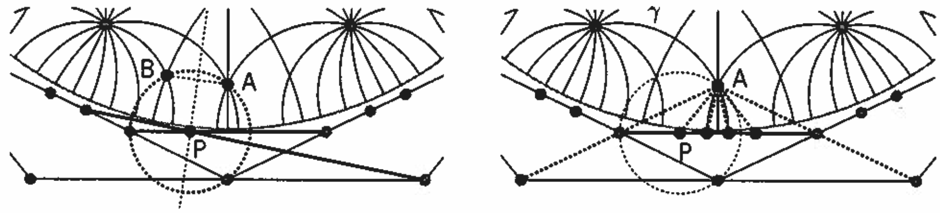

Figure 11: Efficiently producing a geodesic through two points and rotating a geodesic about a point

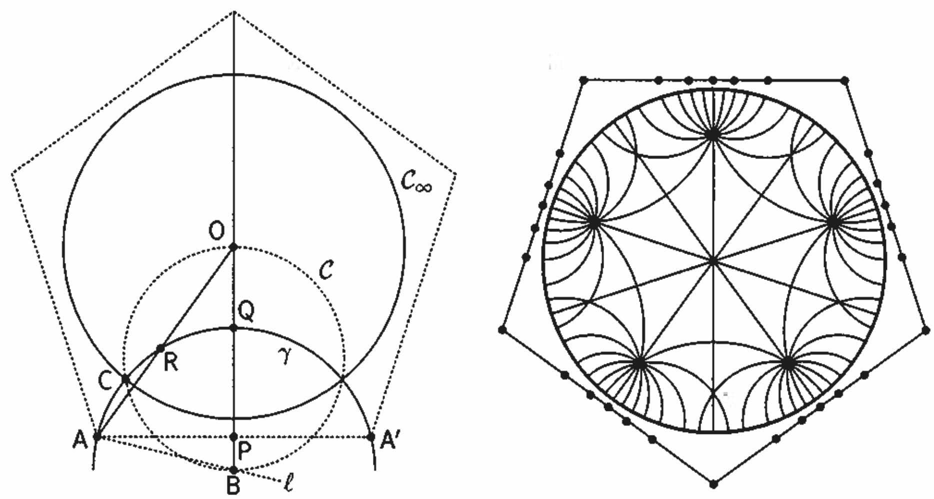

Figure 12: Constructing a $p, q, 2$ triangle

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