Fuzzy *-congruences on regular *-semigroups

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Abstract: The aim of this paper is to describe the fuzzy *-congruences on a regular *-semigroup in terms of their fuzzy projection kernels and fuzzy projection traces.

Keywords: Regular *-semigroup; fuzzy *-congruence; fuzzy *-congruence pair; fuzzy projection kernel; fuzzy projection trace.

1 Introduction

A semigroup S with a unary operation $*: S \to S$ is called a regular *-semigroup if it satisfies

(1) $(x^*)^* = x$;

(2) $(xy)^* = y^*x^*$;

(3) $xx^*x = x$.

Let S be an inverse semigroup. It is easy to see that S is a regular *-semigroup with respect to the unary operation $*: S \to S$ defined by $x^* = x^{-1}$. However, a regular *-semigroup is not necessary to be an inverse semigroup(to see [5]).

In [1], Al-Thukair studied fuzzy congruences on inverse semigroups by introducing the fuzzy analogs of the kernel and trace of a congruence. In this paper, we utilize the type of algebraic techniques to carry on the investigation of fuzzy *-congruences on regular *-semigroups. Firstly, we introduce fuzzy projection kernel and projection trace in Definition 2.1 and fuzzy *-congruence on a regular *-semigroup S (see Definition 3.1), and characterize the largest one of fuzzy *-congruences on S with the same fuzzy projection trace in Proposition 3.3. Secondly, we introduce fuzzy *-congruence pairs and prove in Lemma 4.5 that each pair defines a fuzzy *-congruence. Finally, we prove in Theorem 4.6 that there is a one-one correspondence between fuzzy *-congruence on a regular semigroup and fuzzy *-congruence pairs.

2 Preliminaries

In this paper, the unit [0, 1] is denoted by I, S will mean a regular *-semigroup, and E_S the set of all idempotents in S. An idempotent e of S is called a projection if $e^* = e$. Denote the set of projections in S by P_S. All fuzzy sets of a set X are maps $\mu : X \to I$, and all fuzzy relations are maps $\theta : X \times X \to I$. For $x, y \in I, x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$. If θ and φ are two fuzzy relations on a set X, then $\theta \leq \varphi$ means that $\theta(x, y) \leq \varphi(x, y)$ for any $x, y \in X$. Suppose that θ and φ are two fuzzy relations on a set X. Then their composition denoted by $\theta \circ \varphi$ and defined as

$$\theta \circ \varphi(x, y) = \bigvee_{z \in X} \theta(x, z) \wedge \varphi(z, y) \quad \forall x, y \in X.$$

A fuzzy relation θ on a set X is said to be a fuzzy equivalent relation on X if

(1) $\theta(x, x) = 1 \quad \forall x \in X$,

(2) $\theta(x, y) = \theta(y, x) \quad \forall x, y \in X$,

(3) $\theta \circ \theta \leq \theta$.

Let θ be a fuzzy relation on a regular *-semigroup S. θ is called fuzzy left [right] compatible if $\theta(x, y) \leq \theta(zx, zy)[\theta(x, y) \leq \theta(xz, yz)]$ for all $x, y, z \in S$. It is called fuzzy compatible if $\theta(x, y) \wedge \theta(z, t) \leq \theta(xz, yt)$ for all $x, y, z, t \in S$. A fuzzy relation on S is a fuzzy left congruence[right congruence, congruence] if it is left compatible[right compatible, compatible]. If θ is a fuzzy equivalence relation on S, it is easy to see that θ is a fuzzy congruence iff θ is fuzzy left and right compatible.

Definition 2.1. Let θ be a congruence on S. We denote the fuzzy projection kernel of θ by K_θ and is defined as $K_\theta(x) = \bigvee_{e \in P_S} \theta(x, e) \quad \forall x \in S$. The fuzzy projection trace of θ denoted T_θ and is defined as $T_\theta(e, f) = \theta(e, f) \quad \forall e, f \in P_S$.

The following proposition states well-known results which are used frequently in this paper.

Proposition 2.2 [4]. *Let S be a regular *-semigroup. Then*

(1) $E_S = P_S^2$,

(2) $a^*ea \in P_S$ *for any* $a \in S$ *and* $e \in P_S$.

3 Fuzzy *-congruences

Definition 3.1. Let θ be a fuzzy congruence on S. θ is called a fuzzy *-congruence if $\theta(x^*, y^*) = \theta(x, y) \ \ \forall x, y \in S$.

The following example illustrates that a fuzzy congruence is not necessary to be a fuzzy *-congruence.

Example 3.2. Suppose S be a regular *-semigroup such that there is a congruence ρ on S which is not a *-congruence (see [5]). Define a fuzzy relation θ on S by

$$\theta(x,y) = \begin{cases} 1 & if \quad (x,y) \in \rho \\ 0 & if \quad (x,y) \notin \rho. \end{cases}$$

It is easy to see that θ is a fuzzy congruence on S. Notice that ρ is not a *-congruence on S. So there exist $a, b \in S$ such that $(a, b) \in \rho$, but $(a^*, b^*) \notin \rho$. Thus, we have $\theta(a, b) \neq \theta(a^*, b^*)$. Therefore, θ is not a fuzzy *-congruence.

Proposition 3.3. *Let S be a regular *-semigroup, θ a fuzzy *-congruence, and [θ] the set of fuzzy *-congruences on S which are of the same projection trace as θ. Then the fuzzy relation θ_{max} defined by* $\theta_{max}(a, b) = \bigwedge_{e \in P_S} \theta(aea^*, beb^*) \wedge \theta(a^*ea, b^*eb) \quad \forall a, b \in S$ *is the largest one of [θ].*

Proof. It is immediate that θ_{max} is a fuzzy equivalence relation on S. To show it is fuzzy right compatible, we let $a, b, c \in S$. Then

$$\theta_{max}(ac, bc) = \bigwedge_{e \in P_S} \theta(acec^*a^*, bcec^*b^*) \wedge \theta(c^*a^*eac, c^*b^*ebc)$$

$\geq \bigwedge_{e \in P_S} \theta(aea^*, beb^*) \wedge \theta(a^*ea, b^*eb)$

(because $cP_Sc^* \subseteq P_S$ and θ is a fuzzy congruence)

$= \theta_{max}(a, b)$.

Similarly, we can show that θ_{max} is fuzzy left compatible. Therefore, θ_{max} is a fuzzy congruence on S. It is straightforward that θ_{max} is a fuzzy *-congruence on S. Now let $e, f \in P_S$. Then $\theta(epe^*, fpf^*) = \theta(e^*pe, f^*pf) = \theta(epe, fpf) \geq \theta(e, f) \wedge \theta(p, p) \wedge \theta(e, f) = \theta(e, f)$ for all $p \in P_S$. Hence $\theta_{max}(e, f) \geq \theta(e, f)$. On the other hand, since $\theta(e, f) \geq \theta(e, fef) \wedge \theta(fef, efe) \wedge \theta(efe, f)$ and $\theta(fef, efe) = \theta(f(fef), (efe)e) \geq \theta(f, efe) \wedge \theta(fef, e) = \theta(e, fef) \wedge \theta(efe, f)$, we have $\theta(e, f) \geq \theta(e, fef) \wedge \theta(efe, f) = \theta(eee^*, fef^*) \wedge \theta(efe^*, fff^*) \geq \theta_{max}(e, f)$. So $\theta_{max}(e, f) = \theta(e, f)$. Thus $\theta_{max} \in [\theta]$. Finally, suppose that φ is any fuzzy *-congruence on S such that $\varphi \in [\theta]$. Let $a, b \in S$. Then $\varphi(a, b) = \varphi(a, b) \wedge \varphi(e, e) \wedge \varphi(a^*, b^*) \leq \varphi(aea^*, beb^*) = \theta(aea^*, beb^*)$ for all $e \in P_S$. Similarly, we have $\varphi(a, b) \leq \theta(a^*ea, b^*eb)$ for all $e \in P_S$. Hence, $\varphi(a, b) \leq \theta_{max}(a, b) \quad \forall a, b \in S$, that is, $\varphi \leq \theta_{max}$.

4 Fuzzy *-congruence pairs

Definition 4.1. A fuzzy subset K of S is called fuzzy normal if it satisfies the following:

(1) $K(e) = 1 \quad \forall e \in P_S$(i.e. K is fuzzy p-full),

(2) $K(s^*xs) \geq K(x) \quad \forall x, s \in S$(i.e. K is fuzzy *-self conjugate),

(3) $K(x^*) = K(x) \quad \forall x \in S$(i.e. K is a fuzzy *-subset).

Definition 4.2. A fuzzy equivalence relation τ on P_S is called a fuzzy normal equivalence relation on P_S if $\tau(s^*es, s^*fs) \geq \tau(e, f) \quad \forall e, f \in P_S, s \in S$.

Let τ be a fuzzy normal equivalence relation on P_S and K a fuzzy normal subset of S. The pair (K, τ) is called fuzzy *-congruence pair for S, if it satisfies

(C1) $\tau(aa^*, a^*a) \geq K(a)$,

(C2) $K(ab) \geq K(aeb) \wedge \tau(e, a^*a)$,

(C3) $ef \in P$ implies $K(ab) \geq K(a) \wedge K(b) \wedge \tau(a^*a, e) \wedge (bb^*, f)$,
for any $a, b \in S$ and $e, f \in P$. In this case, define a fuzzy relation $\mu_{(K,\tau)}$ on S by

$$\mu_{(K,\tau)}(a,b) = \tau(a^*a, b^*b) \wedge \tau(aa^*, bb^*) \wedge K(ab^*) \wedge K(a^*b).$$

Lemma 4.3. *Let (K, τ) be a fuzzy *-congruence pair for S. Then we have*

(1) $K(a) \geq K(ae) \wedge \tau(e, a^*a)$,

(2) $K(aeb^*) \wedge \tau(aea^*, beb^*) \geq K(ab^*) \wedge K(a^*b) \wedge \tau(a^*a, b^*b) \wedge \tau(aa^*, bb^*)$,
for any $a, b \in S$ and $e \in P_S$.

Proof. (1) For any $a \in S, e \in P_S$, we have

$K(a) = K(aa^*a)$

$\geq K(aea^*a) \wedge \tau(e, a^*a)$ (by (C2))

$\geq K(ae) \wedge K(a^*a) \wedge \tau((ae)^*(ae), e) \wedge \tau((a^*a)(a^*a)^*, e) \wedge \tau(e, a^*a)$ (by (C3))

$= K(ae) \wedge \tau(e^*a^*ae, e^*ee) \wedge \tau(e, a^*a)$

(because $K(a^*a) = 1$ and $(a^*a)(a^*a)^* = a^*a$)

$= K(ae) \wedge \tau(a^*a, e)$ (by Definition 4.2).

(2) Note first that $bb^*, beb^* \in P_S$ and $(bb^*)(beb^*) = beb^* \in P_S$. We have

$K(aeb^*) \geq K(ab^*beb^*) \wedge \tau(b^*b, a^*a)$ (by (C2))

$\geq K(ab^*) \wedge K(beb^*) \wedge \tau((ab^*)^*(ab^*), bb^*) \wedge \tau((beb^*)(beb^*)^*, beb^*) \wedge \tau(b^*b, a^*a)$

(by (C3))

$= K(ab^*) \wedge \tau(ba^*ab^*, bb^*) \wedge \tau(b^*b, a^*a)$

(because $K(beb^*) = 1$ and $\tau((beb^*)(beb^*)^*, beb^*) = \tau(beb^*, beb^*) = 1$).
Since τ is fuzzy normal, we have $\tau(ba^*ab^*, bb^*) = \tau(ba^*ab^*, bb^*bb^*) \geq \tau(a^*a, b^*b)$.
Thus,

$$K(aeb^*) \geq K(ab^*) \wedge \tau(a^*a, b^*b). \tag{1}$$

Also because τ is fuzzy normal and by (C1), we have

$$\tau(aea^*, aea^*bb^*aea^*) = \tau(aea^*(aa^*)aea^*, aea^*(bb^*)aea^*) \geq \tau(aa^*, bb^*)$$

and

$$\tau(aea^*bb^*aea^*, aeb^*aa^*bea^*) \geq \tau(a^*bb^*a, b^*aa^*b) \geq K(a^*b).$$

Hence

$$\tau(aea^*, aeb^*aa^*bea^*) \geq \tau(aea^*, aea^*bb^*aea^*) \wedge \tau(aea^*bb^*aea^*, aeb^*aa^*bea^*)$$

$$\geq \tau(aa^*, bb^*) \wedge K(a^*b). \quad (2)$$

Since

$$\tau(aeb^*aa^*bea^*, aeb^*bea^*) \geq \tau(b^*aa^*b, b^*b) = \tau(b^*aa^*b, b^*bb^*b) \geq \tau(aa^*, bb^*),$$

$$\tau(aeb^*bea^*, bea^*aeb^*) = \tau(aeb^*(aeb^*)^*, (aeb^*)^*aeb^*) \geq K(aeb^*),$$

and

$$\tau(bea^*aeb^*, beb^*) = \tau(bea^*aeb^*, beb^*beb^*) \geq \tau(a^*a, b^*b),$$

we have

$$\tau(aeb^*aa^*bea^*, beb^*)$$

$$\geq \tau(aeb^*aa^*bea^*, aeb^*bea^*) \wedge \tau(aeb^*bea^*, bea^*aeb^*) \wedge \tau(bea^*aeb^*, beb^*)$$

$$\geq \tau(aa^*, bb^*) \wedge K(aeb^*) \wedge \tau(a^*a, b^*b) \quad (3)$$

Now, from (2) and (3), we get

$$\tau(aea^*, beb^*) \geq \tau(aea^*, aeb^*aa^*bea^*) \wedge \tau(aeb^*aa^*bea^*, beb^*)$$

$$\geq \tau(aa^*, bb^*) \wedge K(a^*b) \wedge K(aeb^*) \wedge \tau(a^*a, b^*b). \quad (4)$$

Hence

$$K(aeb^*) \wedge \tau(aea^*, beb^*) \geq K(ab^*) \wedge K(a^*b) \wedge \tau(aa^*, bb^*) \wedge \tau(a^*a, b^*b)$$

by using (1) and (4) above.

Lemma 4.4. *If θ is a fuzzy *-congruence on S, then (K_θ, T_θ) is a fuzzy *-congruence pair for S.*

Proof. (1) It is clear that $K_\theta(e) = 1$ and $K_\theta(a^*) = K_\theta(a)$ for all $e \in P_S, a \in S$. Also

$$K_\theta(s^*xs) = \bigvee_{e \in P_S} \theta(s^*xs, e)$$

$$\geq \bigvee_{e\in P_S} \theta(s^*xs, s^*es) \qquad \text{(because } s^*P_Ss \subseteq P_S\text{)}$$

$$\geq \bigvee_{e\in P_S} \theta(x, e) \qquad \text{(because } \theta \text{ is a fuzzy congruence).}$$

This proves that K_θ is a fuzzy normal subset of S.

(2) $T_\theta(s^*es, s^*fs) = \theta(s^*es, s^*fs) \geq \theta(e, f)$, since θ is a congruence, and $\theta(e, f) = T_\theta(e, f) \quad \forall e, f \in P_S, s \in S$. Thus, T_θ is a fuzzy normal equivalence relation on P_S.

(3) For all $e \in P_S$, we have

$$T_\theta(aa^*, a^*a) = \theta(aa^*, a^*a)$$

$$\geq \theta(aa^*, e) \wedge \theta(e, a^*a) = \theta(aa^*, ee) \wedge \theta(ee, a^*a)$$

$$\geq \theta(a, e) \wedge \theta(a^*, e) \wedge \theta(e, a^*) \wedge \theta(e, a) = \theta(a, e) \wedge \theta(a^*, e)$$

$$= \theta(a, e) \qquad \text{(because } \theta \text{ is a fuzzy *-congruence).}$$

Thus, $T_\theta(aa^*, a^*a) \geq K_\theta(a)$ for all $a \in S$.

(4) Since

$$K_\theta(ab) \geq \theta(ab, e)$$

$$\geq \theta(ab, ae_0b) \wedge \theta(ae_0b, e)$$

$$\geq \theta(a, ae_0) \wedge \theta(ae_0b, e)$$

$$= \theta(aa^*a, ae_0) \wedge \theta(ae_0b, e)$$

$$\geq \theta(a^*a, e_0) \wedge \theta(ae_0b, e)$$

$$= \theta(ae_0b, e) \wedge T_\theta(e_0, a^*a),$$

it follows that $K_\theta(ab) \geq (\bigvee_{e\in P_S} \theta(ae_0b, e)) \wedge T_\theta(e_0, a^*a)$, and thus,

$$K_\theta(ab) \geq K_\theta(ae_0b) \wedge T_\theta(e_0, a^*a)$$

for any $a, b \in S$ and $e_0 \in P_S$.

(5) Let $e, f \in P_S$ such that $ef \in P_S$. For any $p, q \in P_S$, we have that $\theta(a, p) = \theta(a^*, p)$ and $\theta(b, q) = \theta(b^*, q)$, and thus,

$$\theta(a, p) = \theta(a^*, p) \wedge \theta(a, p) \leq \theta(a^*a, p)$$

and

$$\theta(b, q) = \theta(b, q) \wedge \theta(b^*, q) \leq \theta(bb^*, q).$$

Therefore,

$$K_\theta(ab) \geq \theta(ab, ef)$$

$$\geq \theta(a,e) \wedge \theta(b,f)$$

$$\geq \theta(a,p) \wedge \theta(p,a^*a) \wedge \theta(a^*a,e) \wedge \theta(b,q) \wedge \theta(q,bb^*) \wedge \theta(bb^*,f)$$

$$= \theta(a,p) \wedge \theta(a^*a,e) \wedge \theta(b,q) \wedge \theta(bb^*,f).$$

Taking the sup over $p, q \in P_S$ independently, we get

$$K_\theta(ab) \geq K_\theta(a) \wedge K_\theta(b) \wedge T_\theta(a^*a, e) \wedge T_\theta(bb^*, f).$$

This completes the proof.

Lemma 4.5. *If (K, τ) is a fuzzy *-congruence pair for S, then $\mu_{(K,\tau)}$ is a fuzzy *-congruence on S.*

Proof. Let $\theta = \mu_{(K,\tau)}$. For $a, b, c \in S$.

(1) Since K is fuzzy p-full and $aa^*, a^*a \in P_S$, we have $K(aa^*) = K(a^*a) = 1$. So

$$\theta(a,a) = \tau(aa^*, aa^*) \wedge \tau(a^*a, a^*a) \wedge K(aa^*) \wedge K(a^*a) = 1.$$

(2) Since K is a fuzzy *-subset of S, then $K(ba^*) = K((ab^*)^*) = K(ab^*)$ and $K(b^*a) = K((a^*b)^*) = K(a^*b)$. Thus,

$$\theta(b,a) = \tau(bb^*, aa^*) \wedge \tau(b^*b, a^*a) \wedge K(ba^*) \wedge K(b^*a)$$

$$= \tau(aa^*, bb^*) \wedge \tau(a^*a, b^*b) \wedge K(ab^*) \wedge K(a^*b) = \theta(a,b).$$

(3) Notice that $\tau(aa^*, cc^*) \geq \tau(aa^*, bb^*) \wedge \tau(bb^*, cc^*)$ and $\tau(a^*a, c^*c) \geq \tau(a^*a, b^*b) \wedge \tau(b^*b, c^*c)$. By (C2) and (C3), and because τ is fuzzy normal, we have

$$K(ac^*) \geq K(a(b^*b)c^*) \wedge \tau(b^*b, a^*a)$$

$$\geq K(ab^*) \wedge K(bc^*) \wedge \tau((ab^*)^*ab^*, bb^*) \wedge \tau(bc^*(bc^*)^*, bb^*) \wedge \tau(b^*b, a^*a)$$

$$= K(ab^*) \wedge K(bc^*) \wedge \tau(b(a^*a)b^*, b(b^*b)b^*) \wedge \tau(b(c^*c)b^*, b(b^*b)b^*) \wedge \tau(b^*b, a^*a)$$

$$\geq K(ab^*) \wedge K(bc^*) \wedge \tau(a^*a, b^*b) \wedge \tau(b^*b, c^*c).$$

Similarly, we have $K(a^*c) \geq K(a^*b) \wedge K(b^*c) \wedge \tau(aa^*, bb^*) \wedge \tau(bb^*, cc^*)$. Thus, it follows that $\theta(a,c) \geq \theta(a,b) \wedge \theta(b,c)$. Therefore, $\theta \circ \theta \leq \theta$.

(4) Since τ is fuzzy normal and K is fuzzy *-self conjugate, we have $K((ac)^*bc) \wedge \tau((ac)^*ac, (bc)^*bc) = K(c^*(a^*b)c) \wedge \tau(c^*(a^*a)c, c^*(b^*b)c) \geq K(a^*b) \wedge \tau(a^*a, b^*b)$, and by Lemma 4.3(2), we have

$$K(ac(bc)^*) \wedge \tau(ac(ac)^*, bc(bc)^*) = K(a(cc^*)b^*) \wedge \tau(a(cc^*)a^*, b(cc^*)b^*)$$

$$\geq K(ab^*) \wedge K(a^*b) \wedge \tau(aa^*, bb^*) \wedge \tau(a^*a, b^*b)$$

Thus,

$$\theta(ac, bc) = \tau(ac(ac)^*, bc(bc)^*) \wedge \tau((ac)^*ac, (bc)^*bc) \wedge K(ac(bc)^*) \wedge K((ac)^*bc)$$

$$\geq \tau(aa^*, bb^*) \wedge \tau(a^*a, b^*b) \wedge K(ab^*) \wedge K(a^*b) = \theta(a, b).$$

Similarly, we have $\theta(ca, cb) \geq \theta(a, b)$.

(5) $\theta(a^*, b^*) = \theta(a, b)$ is clear.

From (1), (2), (3), (4) and (5) we conclude that θ is a fuzzy *-congruence on S.

The following theorem is our main result in this paper.

Theorem 4.6. *Let S be a regular *-semigroup. If (K, τ) is a fuzzy *-congruence pair for S, then $\theta = \mu_{(K,\tau)}$ is the unique fuzzy *-congruence on S for which $K_\theta = K$ and $T_\theta = \tau$. Conversely, if μ is a fuzzy *-congruence on S, then (K_μ, T_μ) is a fuzzy *-congruence pair for S and $\mu_{(K_\mu,T_\mu)} = \mu$.*

Proof. Let (K, τ) is a fuzzy *-congruence pair for S. Then, by Lemma 4.5, $\theta = \mu_{(K,\tau)}$ is a fuzzy *-congruence on S. We now wish to show that

$$K_\theta = K \text{ and } T_\theta = \tau.$$

To show that $K_\theta = K$, let $x \in S$, then $K_\theta(x) = \bigvee_{e \in P_S} \theta(x, e)$. But,

$$\theta(x, e) = \tau(xx^*, ee^*) \wedge \tau(x^*x, e^*e) \wedge K(xe^*) \wedge K(x^*e)$$

$$= \tau(xx^*, e) \wedge \tau(x^*x, e) \wedge K(xe) \wedge K(x^*e)$$

$$\leq K(xe) \wedge \tau(e, x^*x) \leq K(x) \qquad \text{by Lemma 4.3(1).}$$

Thus, $K_\theta(x) \leq K(x)$.

On the other hand, by (C1) and definition 4.1(2) and (3), we have

$$K(x) = \tau(xx^*, x^*x) \wedge \tau(x^*x, x^*x) \wedge K(x) \wedge K(x^*x^*x)$$

$$= \tau(xx^*, x^*x(x^*x)^*) \wedge \tau(x^*x, (x^*x)^*x^*x) \wedge K(x(x^*x)^*) \wedge K(x^*(x^*x))$$
$$= \theta(x, x^*x) \leq K_\theta(x).$$

Therefore $K_\theta = K$.

To prove $T_\theta = \tau$, we let $e, f \in P_S$. By (C3), we have $K(ef) \geq K(e) \wedge K(f) \wedge \tau(e^*e, f) \wedge \tau(ff^*, f) = \tau(e, f)$. Thus,

$$T_\theta(e, f) = \theta(e, f) = \tau(ee^*, ff^*) \wedge \tau(e^*e, f^*f) \wedge K(ef^*) \wedge K(e^*f)$$
$$= \tau(e, f) \wedge K(ef) = \tau(e, f).$$

To prove uniqueness let φ be a fuzzy *-congruence on S such that $K_\varphi = K$ and $T_\varphi = \tau$. We wish to show that $\varphi = \theta$. Since

$$\varphi(a, b) = \varphi(a, b) \wedge \varphi(a^*, b^*) \leq \varphi(aa^*, bb^*) = \tau(aa^*, bb^*).$$

and

$$\varphi(a, b) \leq \varphi(ab^*, bb^*) \leq \bigvee_{e \in P_S} \varphi(ab^*, e) = K_\varphi(ab^*) = K(ab^*),$$

and similarly, we have that $\varphi(a, b) \leq \tau(a^*a, b^*b)$ and $\varphi(a, b) \leq K(a^*b)$, then

$$\theta(a, b) = \tau(aa^*, bb^*) \wedge \tau(a^*a, b^*b) \wedge K(ab^*) \wedge K(a^*b) \geq \varphi(a, b).$$

On the other hand, since $\varphi(a, eb) \geq \varphi(a, ab^*b) \wedge \varphi(ab^*b, eb) = \varphi(aa^*a, ab^*b) \wedge \varphi(ab^*b, eb) \geq \varphi(a^*a, b^*b) \wedge \varphi(ab^*, e)$ and $\varphi(ea, b) \geq \varphi(ea, ba^*a) \wedge \varphi(ba^*a, b) \geq \varphi(e, ba^*) \wedge \varphi(a^*a, b^*b) = \varphi(e^*, (ba^*)^*) \wedge \varphi(a^*a, b^*b) = \varphi(e, ab^*) \wedge \varphi(a^*a, b^*b)$, then

$$\varphi(a, b) \geq \varphi(a, eb) \wedge \varphi(eb, ea) \wedge \varphi(ea, b)$$

$$= \varphi(a, eb) \wedge \varphi(ea, b) \qquad \text{(because } \varphi(eb, ea) = \varphi(eeb, ea) \geq \varphi(eb, a)\text{)}$$

$$\geq \varphi(a^*a, b^*b) \wedge \varphi(ab^*, e).$$

But e is arbitrary in P_S. Hence

$$\varphi(a, b) \geq \varphi(a^*a, b^*b) \wedge \bigvee_{e \in P_S} \varphi(ab^*, e)$$

$$= \tau(a^*a, b^*b) \wedge K_\varphi(ab^*) = \tau(a^*a, b^*b) \wedge K(ab^*)$$
$$\geq \tau(aa^*, bb^*) \wedge \tau(a^*a, b^*b) \wedge K(ab^*) \wedge K(a^*b) = \theta(a, b).$$

Therefor, $\varphi = \theta$.

Conversely, let μ be a fuzzy *-congruence on S. Then by Lemma 4.4, (K_μ, T_μ) is a fuzzy *-congruence pair for S. So we need only to show that $\mu_{(K_\mu, T_\mu)} = \mu$, but this follows from the uniqueness argument proved earlier in the proof of this theorem.

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