

**Going in circles:
variations on the Money-Coutts theorem**

by

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Given a triangle A_1, A_2, A_3 , consider the following chain of circles: S_1 is inscribed in the angle A_1 ; S_2 inscribed in the angle A_2 and tangent to S_1 ; S_3 inscribed in the angle A_3 and tangent to S_2 ; S_4 inscribed in the angle A_1 and tangent to S_3 ; etc. (each circle S_i is the smaller of the two circles tangent to S_{i-1}). It is a lesser known elementary geometry gem that this process is 6-periodic: $S_7 = S_1$ – see figure 1. Moreover, the 6-periodicity persists if the sides of the triangle are replaced by arcs of circles. This result, known as the Money-Coutts theorem, was published as late as in 1971 – see [T-P], [E], [B-B] for precise formulations and proofs.

Fig. 1

One wonders whether the Money-Coutts theorem extends to other polygons. For example, if P is a regular n -gon (whose vertices are numbered cyclically) then, by symmetry, the circles S_{i-1} and S_{i+1} are congruent for all $i = 1, \dots, n$. It follows that for a regular n -gon the process is $2n$ -periodic if n is odd and n -periodic if n is even – see figure 2.

Fig. 2

The next, Mathematica generated, figure 3 shows that this periodicity is destroyed by a generic perturbation of a regular polygon.

Fig. 3

However the periodicity persists for a certain class of polygons. Let A_1, \dots, A_n be the vertices of a convex n -gon P , let $2\alpha_i$ be the interior angle at A_i , and let $a_i = |A_i A_{i+1}|$. As before, inscribe a circle S_1 in the angle A_1 , then inscribe a circle S_2 in the angle A_2 , tangent to S_1 , and continue cyclically. Let O_i be the center of S_i and r_i its radius. To fix the choice of the circles at each step, assume that the orthogonal projections B_i and B_{i+1}

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of O_i and O_{i+1} on the line $A_i A_{i+1}$ fall on the segment $A_i A_{i+1}$, and that B_i is closer to A_i than B_{i+1} – see figure 4.

Fig. 4

The Money-Coutts theorem extends to the following class of n -gons P that, for the lack of a better name, we call *nice*. Assume that $n \geq 5$, and $\alpha_i + \alpha_{i+1} > \pi/2$ for all i . Let D_i be the intersection point of the lines $A_{i-1}A_i$ and $A_{i+1}A_{i+2}$. Consider the excircles of the triangles $A_{i-1}A_iD_{i-1}$ and $A_iA_{i+1}D_i$ that are tangent to the sides A_iD_{i-1} and A_iD_i , respectively. The polygon P is nice if these two circles coincide for all i – see figure 5.

Fig. 5

Theorem 1. *Let P be a nice n -gon.*

- a). *If n is odd then the sequence of the circles S_i is $2n$ -periodic: $S_1 = S_{2n+1}$.*
- b). *If n is even, assume that*

$$\prod_{i=1}^n (\sqrt{1 - \cot \alpha_i \cot \alpha_{i+1}} + 1)^{(-1)^i} = 1; \quad (1)$$

then the sequence of the circles S_i is n -periodic: $S_1 = S_{n+1}$.

Proof. First, we claim that P is nice if and only if there exists a constant $\rho > 0$ such that

$$a_i = \rho^2 (\tan \alpha_i \tan \alpha_{i+1} - 1) \quad (2)$$

for all i (notice that $\tan \alpha_i \tan \alpha_{i+1} > 1$ since $\alpha_i + \alpha_{i+1} > \pi/2$).

Fig. 6

Consider the excircle of the triangle $A_i A_{i+1} D_i$ in figure 6; let r be its radius. From the right triangle OEA_i one has: $|EA_i| = r \cot \alpha_i$. Likewise, from the right triangle OEA_{i+1} one finds: $|EA_{i+1}| = r \tan \alpha_{i+1}$. It follows that $r \tan \alpha_{i+1} = r \cot \alpha_i + a_i$, and therefore

$$\frac{a_i}{\tan \alpha_i \tan \alpha_{i+1} - 1} = \frac{r}{\tan \alpha_i}. \quad (3)$$

Similarly, let r' be the radius of the excircle of the triangle $A_{i-1} A_i D_{i-1}$; then

$$\frac{a_{i-1}}{\tan \alpha_{i-1} \tan \alpha_i - 1} = \frac{r'}{\tan \alpha_i}. \quad (4)$$

The polygon is nice if and only if $r = r'$; combining (3) and (4), this is equivalent to $a_i/(\tan \alpha_i \tan \alpha_{i+1} - 1)$ being independent of i , as claimed.

Consider figure 4 again. One has:

$$B_i B_{i+1} = 2\sqrt{r_i r_{i+1}}, \quad |A_i B_i| = r_i \cot \alpha_i, \quad |A_{i+1} B_{i+1}| = r_{i+1} \cot \alpha_{i+1},$$

therefore

$$r_i \cot \alpha_i + 2\sqrt{r_i r_{i+1}} + r_{i+1} \cot \alpha_{i+1} = a_i. \quad (5)$$

Introduce new variables:

$$u_i = \sqrt{r_i \cot \alpha_i}, \quad e_i = \sqrt{\tan \alpha_i \tan \alpha_{i+1}}, \quad c_i = \sqrt{a_i};$$

note that $e_i > 1$. The equation (5) reads:

$$u_i^2 + 2e_i u_i u_{i+1} + u_{i+1}^2 = c_i^2. \quad (6)$$

This equation can be explicitly solved in hyperbolic trigonometric functions. Namely, from (6) one finds:

$$u_i + e_i u_{i+1} = \sqrt{c_i^2 + (e_i^2 - 1)u_{i+1}^2}, \quad u_{i+1} + e_i u_i = \sqrt{c_i^2 + (e_i^2 - 1)u_i^2},$$

hence (6) can be rewritten as

$$u_i \sqrt{c_i^2 + (e_i^2 - 1)u_{i+1}^2} + u_{i+1} \sqrt{c_i^2 + (e_i^2 - 1)u_i^2} = c_i^2,$$

or, taking (2) into account, as

$$\frac{u_i}{\rho} \sqrt{1 + \frac{u_{i+1}^2}{\rho^2}} + \frac{u_{i+1}}{\rho} \sqrt{1 + \frac{u_i^2}{\rho^2}} = \frac{c_i}{\rho}. \quad (7)$$

Let $x_i = \sinh^{-1}(u_i/\rho)$ where $\sinh^{-1}(y) = \ln(y + \sqrt{1 + y^2})$. Then (7) reads:

$$\sinh x_i \cosh x_{i+1} + \sinh x_{i+1} \cosh x_i = c_i/\rho \quad \text{or} \quad \sinh(x_i + x_{i+1}) = c_i/\rho. \quad (8)$$

Denote the family of circles inscribed in the i -th angle of the polygon P by \mathcal{F}_i . We have a map $T_i : \mathcal{F}_i \rightarrow \mathcal{F}_{i+1}$ that takes S_i to S_{i+1} . One may use x_i as a coordinate in \mathcal{F}_i ; then (8) means that T_i is a reflection:

$$T_i(x_i) = x_{i+1} = \sinh^{-1}(c_i/\rho) - x_i.$$

This implies the statement of the theorem. If n is odd then the return map

$$T_n T_{n-1} \dots T_1 : \mathcal{F}_1 \rightarrow \mathcal{F}_1$$

is a reflection, and its second iteration is the identity, as claimed. If n is even then the return map is the translation

$$x_1 \rightarrow x_1 + \sum (-1)^i \sinh^{-1}(c_i/\rho) \quad (9)$$

In view of (2), the vanishing of the alternating sum in (9) is equivalent to (1), and we are done.

Remarks. 1. Eliminating ρ from equalities (2), one sees that the codimension of the subvariety of nice n -gons in the $2n$ -dimensional space of n -gons is $n - 1$.

2. If $|e_i| < 1$ then equation (6) can be similarly solved in trigonometric functions. Moreover, equation (6) with $|e_i| < 1$ can be considered as the Cosine rule for the triangle with sides u_i, u_{i+1}, c_i . This triangle is inscribed into a circle whose diameter $c_i/\sqrt{1 - e_i^2}$ does not depend on i – see [B-Z]. Likewise, equation (6) with $e_i > 1$ can be considered as Cosine rule in the Lorentz plane with the metric $dy^2 - dx^2$; the respective triangle is inscribed into a pseudo-circle, that is, a hyperbola $x^2 - y^2 = c^2$ with the "radius" c independent of i .

3. An intermediate case of $e_i = 1$ in equation (6) corresponds to the case when the $(i - 1)$ -st and $(i + 1)$ -st sides of the polygon are parallel. Then equation (6) implies that $u_i + u_{i+1} = c_i$. In particular, it follows that if P is a parallelogram then the sequence of circles S_i is not periodic, unless P is a rhombus (in which case $S_5 = S_1$).

4. It would be interesting to interpret equality (1) geometrically; I do not know such an interpretation.

Next, we discuss how to extend Theorem 1 to the case when the sides of the polygon P are made of arcs of circles. We closely follow [B-B], sect. 6 and 9, and some of the arguments below are only outlined.

One starts with complexification and projectivization: a circle is considered as a conic

$$a(x^2 + y^2) + 2bxz + 2cyz + dz^2 = 0$$

in the complex projective plane; the space of circles $\{(a : b : c : d)\}$ is the complex projective space. Given two generic circles C_1 and C_2 , the family of circles, tangent to both of them, consists of two irreducible components; each component is a rational curve. The two components are seen in the real situation as follows: if a circle S is tangent to

C_1 and C_2 then the line through the two tangency points passes through one of the two centers of similitude of C_1 and C_2 ; a choice of a component is determined by a choice of one of these two centers.

Let C_1, C_2, C_3 be circles in general position. For each of the pairs C_1, C_2 and C_2, C_3 choose one of the two families of circles touching both circles; call these families \mathcal{F}_1 and \mathcal{F}_2 . Consider the curve consisting of pairs of circles $(S_1, S_2) \in \mathcal{F}_1 \times \mathcal{F}_2$ such that S_1 is tangent to S_2 . According to [B-B], this curve has two irreducible components that are rational curves of bidegree (1,1) and another irreducible component that is an elliptic curve of bidegree (2,2) – see figure 7 a) for the former and 7 b) for the latter.

Fig. 7

Let C_1, \dots, C_n be circles in general position. For each $i = 1, \dots, n$ we choose one of the two families of circles, tangent to C_i and C_{i+1} ; call the chosen family \mathcal{F}_i . One has elliptic curves $E_i \subset \mathcal{F}_i \times \mathcal{F}_{i+1}$; denote the projections on the factors by π_i and π'_i .

A collection of circles C_1, \dots, C_n , along with a choice of the families \mathcal{F}_i , will be called *nice* if for every i the following holds: the two Apollonius circles for C_{i-1}, C_i, C_{i+1} that lie in \mathcal{F}_i but not in \mathcal{F}_{i-1} coincide with the two Apollonius circles for C_i, C_{i+1}, C_{i+2} that lie in \mathcal{F}_i but not in \mathcal{F}_{i+1} . Figure 8 shows a nice collection of 4 circles (the family \mathcal{F}_i contains the circles that touch and contain both circles C_i and C_{i+1}). The niceness condition on n circles imposes $2n$ relations: each consecutive circle C_i is tangent to two circles determined by the preceding three circles $C_{i-1}, C_{i-2}, C_{i-3}$. It follows that the variety of nice n -tuples of circles is (at least) n -dimensional. Note also that niceness does not impose any restrictions on a triple of circles: given a choice of \mathcal{F}_1 and \mathcal{F}_2 , one can choose \mathcal{F}_3 so that the triple is nice.

Fig. 8

The next result is an extension of the Money-Coutts theorem as formulated in [B-B] and, at the same time, of Theorem 1 a).

Theorem 2. *Let $\{C_i, \mathcal{F}_i\}$ be a nice collection of n circles. Suppose that circles $S_1 \in \mathcal{F}_1, \dots, S_n \in \mathcal{F}_n, S_{n+1} \in \mathcal{F}_1$ are chosen such that $(S_1, S_2) \in E_1, \dots, (S_n, S_{n+1}) \in E_n$. Then it is possible to choose circles $S_{n+2} \in \mathcal{F}_2, \dots, S_{2n} \in \mathcal{F}_n, S_{2n+1} \in \mathcal{F}_1$ such that S_{2n+1} coincides with S_1 .*

Proof. We repeat the proof from [B-B]. Consider the two projections: $\pi'_{i-1} : E_{i-1} \rightarrow \mathcal{F}_i$ and $\pi_i : E_i \rightarrow \mathcal{F}_i$. The branch points of the first projection are the two null-circles of

the pair C_i, C_{i+1} and the two Apollonius circles for C_{i-1}, C_i, C_{i+1} that lie in \mathcal{F}_i but not in \mathcal{F}_{i-1} ; the branch points of the second projection are again the two null-circles of the pair C_i, C_{i+1} and the two Apollonius circles for C_i, C_{i+1}, C_{i+2} that lie in \mathcal{F}_i but not in \mathcal{F}_{i+1} . Due to niceness, the branch points of π'_{i-1} and π_i coincide.

Therefore there are isomorphisms

$$\phi_1 : E_1 \rightarrow E_2, \phi_2 : E_2 \rightarrow E_3, \dots, \phi_n : E_n \rightarrow E_1$$

such that $\pi_{i+1}\phi_i = \pi'_i$ for all i . Identify $E_1 = \dots = E_n := E$ and $\pi'_i = \pi_{i+1}$ by means of ϕ_i , $i = 1, \dots, n-1$. Then ϕ_n becomes an automorphism τ of E that, with a suitable choice of $\phi_1, \dots, \phi_{n-1}$, can be assumed a translation (and not an involution).

Let σ_i be the involution of E that interchanges the preimages of π_i . One has: $S_1 = \pi_1(e)$ for some $e \in E$. Then $S_2 = \pi'_1(\alpha_1 e)$ where α_1 is either the identity or σ_1 . Continuing the same way, one has:

$$S_{n+1} = \pi'_n(\alpha_n \dots \alpha_1 e) = \pi_1(\tau \alpha_n \dots \alpha_1 e)$$

with $\alpha_j \in \{1, \sigma_j\}$.

Similarly, to determine circles S_{n+2}, \dots, S_{2n+1} with $(S_j, S_{j+1}) \in E$, one chooses $\beta_j \in \{1, \sigma_j\}$, $j = 1, \dots, n$ and sets:

$$S_{n+1+j} = \pi'_j(\beta_j \dots \beta_1 \tau \alpha_n \dots \alpha_1 e);$$

in particular,

$$S_{2n+1} = \pi_1(\tau \beta_n \dots \beta_1 \tau \alpha_n \dots \alpha_1 e).$$

The choice is made as follows: $\beta_j = \alpha_j$ for $j = 2, \dots, n$, and β_1 is chosen so that $\beta_n \dots \beta_1$ is an involution. Then one has:

$$\tau \beta_n \dots \beta_1 \tau \alpha_n \dots \alpha_1 = \beta_n \dots \beta_1 \alpha_n \dots \alpha_1 = (\alpha_n \dots \alpha_2 \beta_1)^2 \beta_1 \alpha_1 = \beta_1 \alpha_1.$$

It follows that $S_{2n+1} = \pi_1(\beta_1 \alpha_1 e) = S_1$, and we are done.

Remarks. 1. The niceness condition on polygon P appeared in Theorem 1 as a technical condition needed in the proof, namely, to provide formula (2); the algebro-geometrical approach gives a more conceptual explanation of this condition.

2. Consider the situation of Theorem 1 a): P is a nice n -gon with odd n , the choice of the families \mathcal{F}_i being determined by the choice of the circles S_i in Theorem 1. Using the same notation as above, one has the isomorphisms $\psi_i : E_i \rightarrow E_{i+1}$ that take (S_i, S_{i+1})

to (S_{i+1}, S_{i+2}) . The through automorphism $\psi_n \dots \psi_1 : E_1 \rightarrow E_1$ is an involution of the elliptic curve, and its second iteration is the identity. In particular, it follows that a nice polygon in Theorem 1 a) can be perturbed by replacing its sides by arcs of (sufficiently large) circles that make a nice collection.

3. Unlike the case of an odd n , Theorem 1 b) is not a particular case of Theorem 2: the choice of the circles S_i in the proof of the latter is not the same as that in the formulation of the former. It would be interesting to find a generalization of Theorem 1 b) along the lines of Theorem 2.

4. It is possible, a priori, that the translation τ in the proof of Theorem 2 is the identity. Can such a poristic case occur for odd n ? The answer is negative for $n = 3$ unless the configuration of circles is very degenerate, namely, they belong to a pencil – see [T-P].

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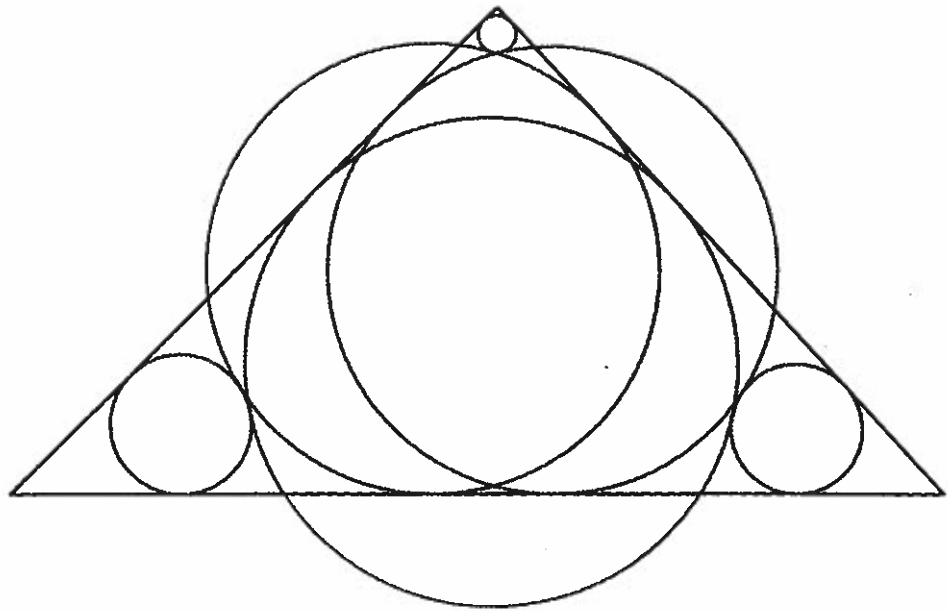


Figure 1

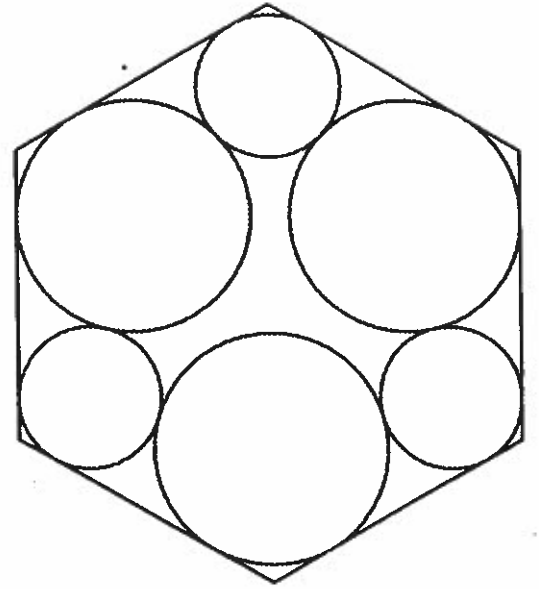
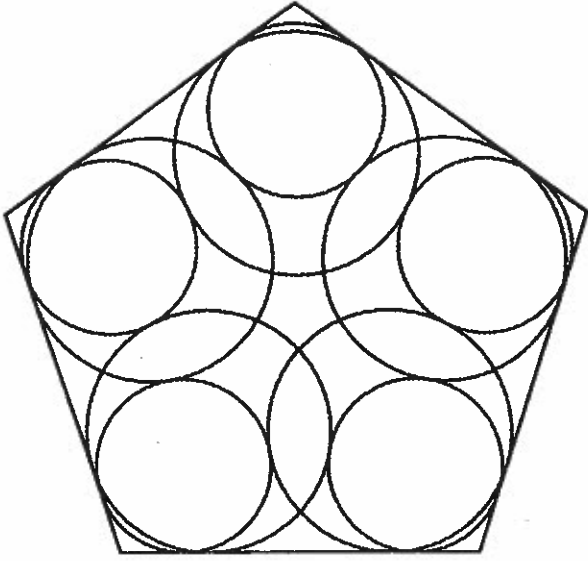


Figure 2

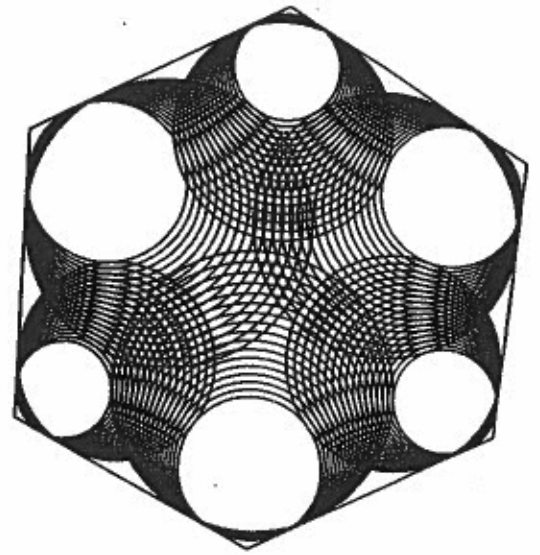
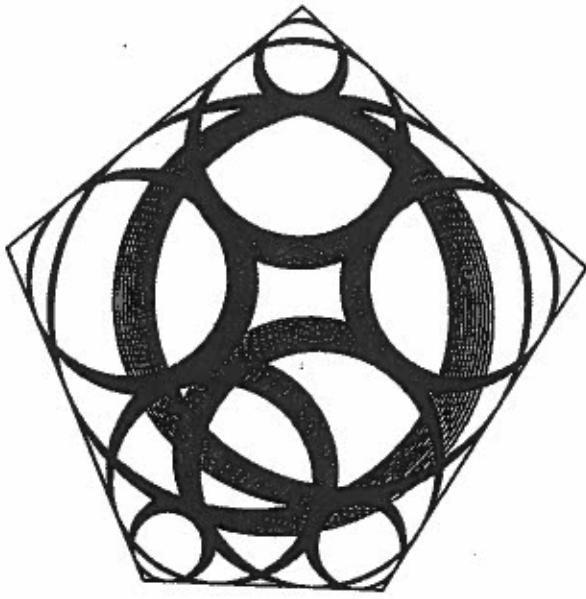


Figure 3

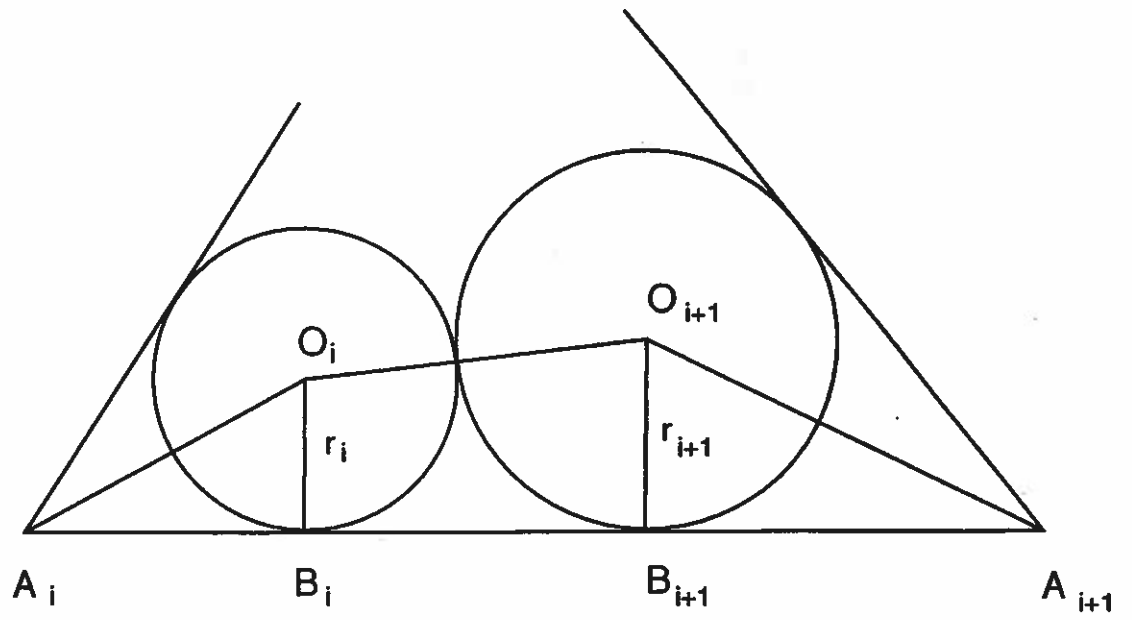


Figure 4

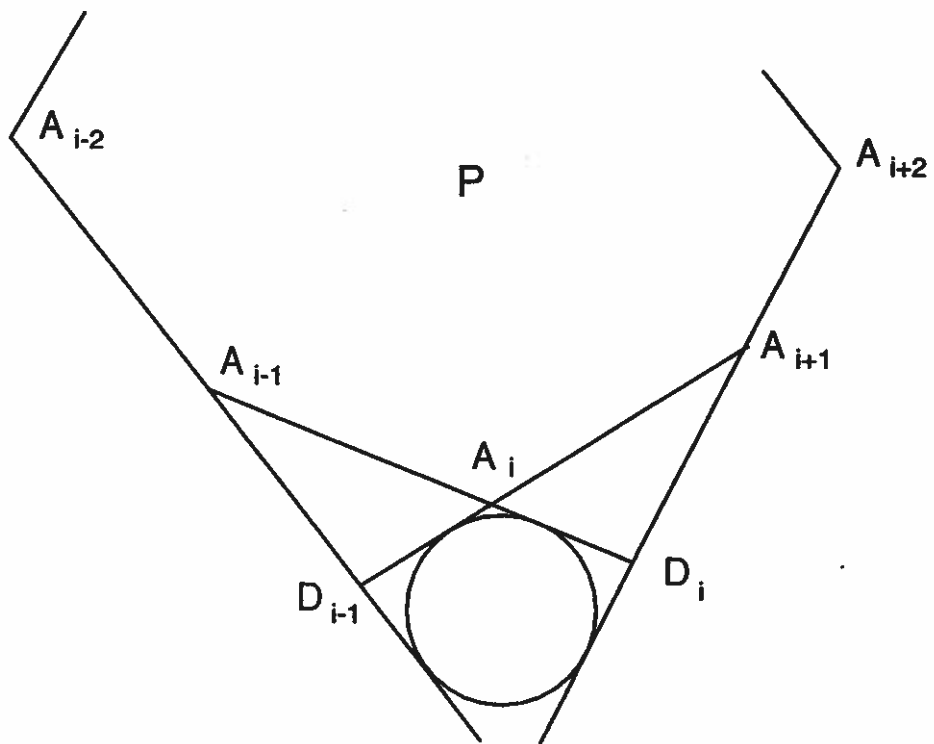


Figure 5

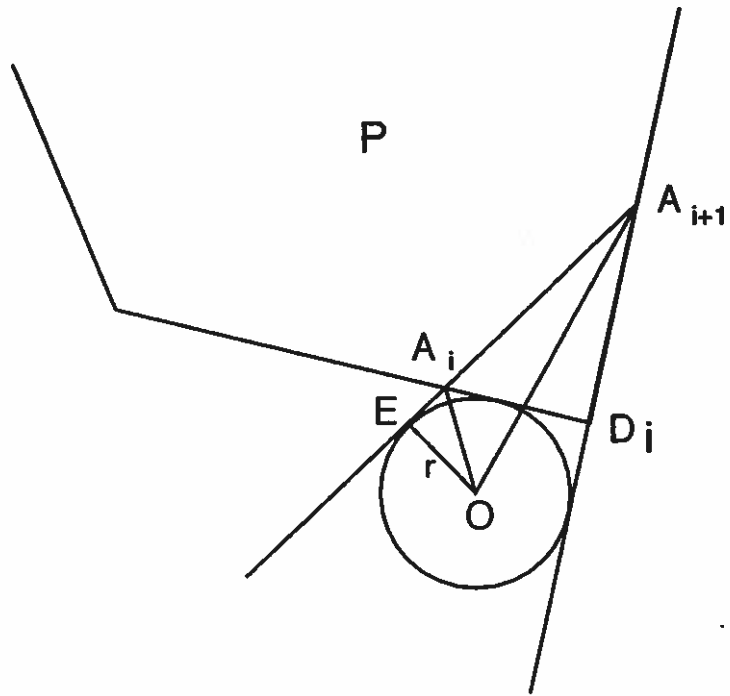


Figure 6

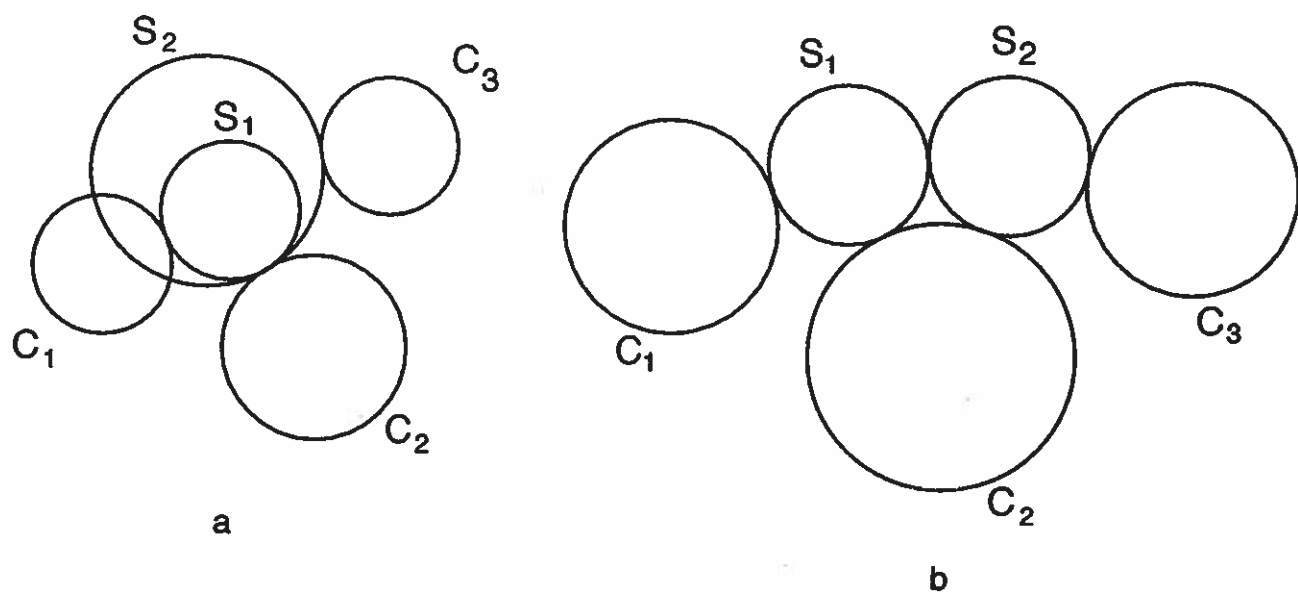


Figure 7

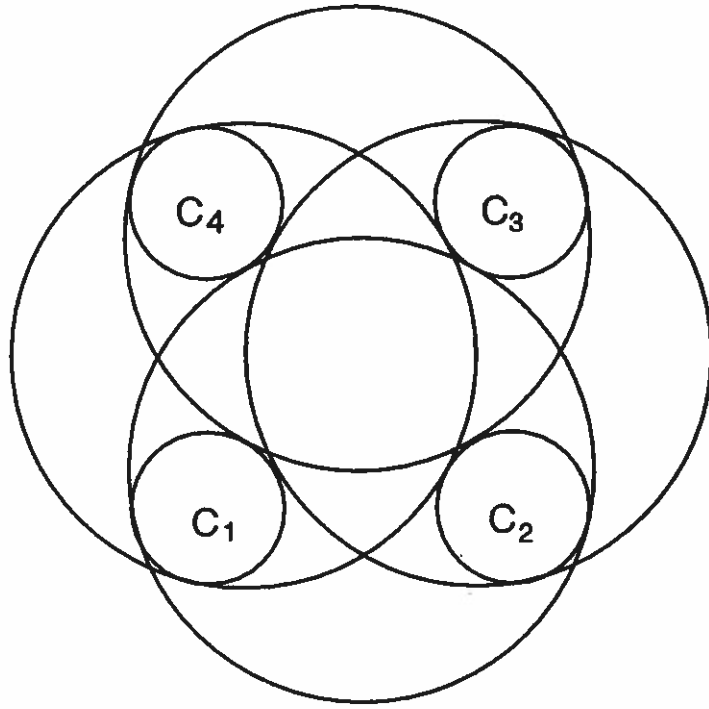


Figure 8

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