Monotonicity principles for singular integral equations in Clifford analysis

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1 Introduction

Monotonicity principles are used to get informations about nonlinear singular integral equations. These results are based on the famous theorem of Browder and Minty (see for example [24])

Theorem 1 (Browder-Minty) *Let X be a real separable Banach space. If the operator $A : X \to X^*$ is monotone, i.e. $\langle Au - Av\,,\, u - v\rangle \geq 0$, coercive, i.e. $\lim\limits_{\|u\|\to\infty} \frac{\langle Au\,,\,u\rangle}{\|u\|} = \infty$, and hemicontinuous then A is surjectiv. If A is moreover strictly monotone, then the solution of $Au = b$, $b \in X^*$, is unique.*

In [21] this theorem is used by L. v. Wolfersdorf to consider singular integral equations on the real halfline involving the singular (complex) Hilbert operator. This theory is extended by Askabarov in [1] to the complex case.
We want to investigate a special family of singular integral operators in Clifford analysis which has in $\mathbb{R}^3$ an application to the nonlinear magnetic field equation ([5]) considered by M. Friedman. Electromagnetic processes are described using quaternionic and Clifford analysis by several authors ([2], [3], [4], [6], [7], [8], [10], [11], [12], [17], [18], [19], [22], [23]). These considerations mainly based on the operator $D + a$. To this subject we want to recommend the book by V.V. Kravchenko and M.V. Shapiro ([13]) and the book by K. Gürlebeck and W. Sprößig ([9]).
Singular integral operators are investigated by S.G. Michlin and S. Prößdorf in [16] and especially using Clifford analytical methods by A. McIntosh, C. Li and S. Semmes in [14] and by A. McIntosh, C. Li and T. Qian in [15].
Properties of the Nemickii operator, monoton operators and so one may be found in the book [24] by E. Zeidler.

2 Clifford algebras

Let us denote by $\mathcal{C}\ell_{0,h}(\mathbb{R})$ the *real Clifford algebra* associated to the Euclidean space $\mathbb{R}^h$ generated by the elements $\{e_j\}_{j=1}^h$ with

$$e_i e_k + e_k e_i = -2\delta_{ik}.$$

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An arbitrary element of $\mathcal{C}\ell_{0,h}(\mathbb{R})$ has a representation of the form

$$a = \sum_I a_I e_I, \qquad a_I \in \mathbb{R},$$

where I denotes ordered l-tuples of the form $I = (i_1, i_2, \ldots, i_l)$, with $1 \leq i_1 < i_2 < \cdots < i_l \leq h$, where $0 \leq l \leq h$. Furthermore, e_I stands for the product $e_{i_1} e_{i_2} \cdots e_{i_l}$. By convention, $e_\emptyset := e_0 := 1$.
On $\mathcal{C}\ell_{0,h}(\mathbb{R})$ may be introduced an involution, sometimes called main involution:

$$\bar{a} := \sum_I a_I \bar{e}_I, \qquad \bar{e}_I := (-1)^{\frac{|I|(|I|+1)}{2}} e_I.$$

The *complexified Clifford algebra* $\mathcal{C}\ell_{0,h}(\mathbb{C})$ associated to $\mathbb{R}^h$ is

$$\mathcal{C}\ell_{0,h} \otimes \mathbb{C},$$

where $\mathbb{C}$ denotes the complex numbers. Thus, an arbitrary element c of the complexified Clifford algebra has a representation of form

$$c = \sum_I c_I e_I = a + ib = \sum_I a_I e_I + i \sum_I b_I e_I, \; c_I = a_I + ib_I \in \mathbb{C}, \; a_I, b_I \in \mathbb{R}.$$

In the complexified Clifford algebra may be introduced another involution:

$$\tilde{c} := \sum_I \bar{c}_I \bar{e}_I,$$

where $\bar{c} = a_I - ib_I$. We call $c_0 e_0 = c_0$ the scalar part of c and denote by $\mathcal{P}$ the operator

$$\mathcal{P} : \mathcal{C}\ell_{0,h}(\mathbb{C}) \to \mathbb{C} : \; \mathcal{P}c = c_0.$$

This operator is a projection because of $\mathcal{P}^2 = \mathcal{P}$ and we denote by $\mathcal{Q}$ the complementary projection $\mathcal{Q} = I - \mathcal{P}$. Further, we denote by $\mathcal{R}e$ the operator

$$\mathcal{R}e \; : \mathcal{C}\ell_{0,h}(\mathbb{C}) \to \mathbb{R} : \mathcal{R}e\, c = a_0.$$

Moreover, the Clifford algebra $\mathcal{C}\ell_{0,h}(\mathbb{C})$ becomes a normed algebra with

$$|c| := \left(\sum_I |c_I|^2 \right)^{\frac{1}{2}} = [c, c].$$

Here,

$$[u, v] := \mathcal{R}e \sum_{I,J} \bar{u}_I \bar{e}_I v_J e_J$$

is a real scalar product and the Clifford algebra $\mathcal{C}\ell_{0,n}$ becomes a real Hilbert space.

3 Clifford analysis

Let G be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary ∂G. This means that ∂G may be covered by finitely many open sets W_k such that each set $G \cap W_k$ is represented by the inequality $x_n > g_k(x_1, x_2, \ldots, x_{n-1})$, where g_k is a Lipschitz function.
On such a domain, the exterior unit normal $n(y)$ is defined for almost all $y \in \partial G$, and Gauß's theorem is valid.
Let $u \in C^1(G)$ (taking values in a Clifford algebra) then the *Dirac operator* is defined as

$$(Du)(x) = \sum_I \sum_{j=1}^n \frac{\partial u_I}{\partial x_j}(x) e_j e_I.$$

Any solution u of the equation $Du = 0$ is called a *left monogenic* function. Analogously, if the Dirac operator is acting from the right any solution u of $uD = 0$ is called a right monogenic function.
Moreover, we consider the *disturbed Dirac operator*

$$D_{ia}u = (D +^{ia} M) = \sum_I \sum_{j=1}^n \frac{\partial u_I}{\partial x_j}(x) e_j e_I + \sum_I \sum_{j=1}^n i a_j u_I e_j e_I + i a_0 \sum_I u_I e_I,$$

where $a = a_0 e_0 + \sum\limits_{j=1}^n a_j e_j$ and $a_k \in \mathbb{R}$, $k = 1, 2, \ldots, n$, thus a is a paravector in the Clifford algebra $\mathcal{C}\ell_{0,n}$.
A fundamental solution of D_{ia} is

$$e_{ia} = e^{-i<a,x>}\{(D -^{ia_0} M) K_{a_0}\},$$

where $< a, x >= \sum\limits_{j=1}^n a_j x_j$ and

$$K_{a_0}(x) = K_{a_0}(|x|) = \frac{1}{(2\pi)^{\frac{n}{2}}} \left(\frac{a_0}{|x|}\right)^{\frac{n}{2}-1} K_{\frac{n}{2}-1}(a_0|x|),$$

is a fundamental solution of $-\Delta + a_0^2$ (cf. [20]) and $K_{\frac{n}{2}-1}$ denotes the modified Bessel functions, the so-called MacDonalds functions, of order $\frac{n}{2} - 1$.
With the aid of this fundamental solution we define

$$T_{ia}u := \int_G e_{ia}(x - y) u(y) dy.$$

Then, we get

$$D_{ia} T_{ia} u = \begin{cases} 0 & \text{in } \mathbb{R}^n \backslash \bar{G} \\ u & \text{in } G. \end{cases}$$

As mentioned above Gauß's formula holds and from this it is derived

$$\int_G D\,u\,dG = \int_\Gamma n\,v\,d\Gamma.$$

Let $L_{2,\mathcal{C}\ell}(G)$ denote the normed space of measurable functions u from G to $\mathcal{C}\ell_{0,h}$ for which the norm

$$||u|| = \left(\int_G |u|^2 dG\right)^{\frac{1}{2}} < \infty,$$

It is a (real!) Hilbert space created by the scalar product

$$(u,v) := \mathcal{R}e \int_G \tilde{u}\,v\,dG = \int_G \mathcal{R}e\,(\tilde{u}\,v)\,dG.$$

4 The Nemyckii operator

We want to study two types of nonsingular integral equations. First, we require the properties of the so-called Nemyckii operator F in a Clifford analysis context. This operator is defined as

$$(Fu)(x) = f(x, u_1(x), u_2(x), \ldots, u_n(x))$$

with $u = (u_1, u_2, \ldots, u_n)$. We make the following assumptions:
(A1) Carathéodory condition: Let $f : G \times \mathbb{R}^n \to \mathcal{C}\ell_{0,n}(\mathbb{C})$ be a given function, where G is a nonempty set in $\mathbb{R}^N$, $n, N \geq 1$. Moreover,

$$x \to f(x,u) \text{ is measurable on } G \text{ for all } u \in \mathbb{R}^n;$$

$$u \to f(x,u) \text{ is continuous on } \mathbb{R}^n \text{ for almost all } x \in G.$$

(A2) Growth condition: For all $(x,u) \in G \times \mathbb{R}^n$,

$$|f(x,u)| \leq a(x) + b|u|.$$

Here, b is a fixed positive number and $a \in L_2(G)$ is a real-valued nonnegative function.

Proposition 1 *Under the two assumptions (A1) and (A2), the following are valid: The Nemyckii operator*

$$F : L_{2,\mathcal{C}\ell}(G) \to L_{2,\mathcal{C}\ell}(G)$$

is continuous and bounded with

$$||Fu||_{L_{2,\alpha}} \leq const(||a||_{L_2} + ||u||_{L_{2,\alpha}})$$

and

$$(Fu, u) = \mathcal{R}e \int_G f(\widetilde{x, u(x)})u(x)dx \quad \textit{for all } u \in L_{2,\mathcal{C}\ell}(G).$$

Moreover,

Monotonicity of f: The function f is monotone with respect to u i.e. $$[f(x,u) - f(x,v), u - v] \geq 0$$ for all $u, v \in L_{2,\mathcal{C}\ell}(G)$.	implies F is monotone.
Strictly monotonicity of f:The function f is strictly monotone with respect to u i.e. $$[f(x,u) - f(x,v), u - v] > 0$$ for all $u, v \in L_{2,\mathcal{C}\ell}(G)$.	implies F is strictly monotone.
Coercivness of f: $$[f(x,u), u] \geq d\|u\|^2 + g(x),$$ where $g \in L_1(G)$.	implies F is coercive and $$(Fu, u) \geq d\|\|u\|\|^2 + \int_G g(x)dx$$ for all $u \in L_{2,\mathcal{C}\ell}(G)$.
Positivity of f: For all $(x,u) \in G \times \mathbb{R}^n$ $$[f(x,u), u] \geq 0.$$	implies $(Fu, u) \geq 0$ for all $u \in L_{2,\mathcal{C}\ell}(G)$.

Asymptotic positivity of f: There exists a number $R > 0$ such that $$[f(x,u),u] \geq 0$$ holds for all $(x,u) \in G \times \mathbb{R}^n$ with $\lvert u\rvert \geq R$ and meas $G < \infty$.	implies $(Fu,u) \geq -c$ for all $u \in L_{2,\alpha}(G)$, where c is a positive constant.
Lipschitz continuity of f: There is a constant $L > 0$ such that $$\lvert f(x,u) - f(x,v)\rvert \leq L\lvert u-v\rvert.$$	implies F is Lipschitz continuous.

5 A family of positive operators

5.1 The singular integral operator $D_{ia}\mathcal{P}T_{ia}$

The aim of this section is to show that the operator $D_{ia}\mathcal{P}T_{ia}$ is a (general) singular integral operator of Calderon-Zygmund-type. Here, "general" means a singular operator plus weakly singular operator parts.
For this purpose it is usefull to recall some properties of Bessel functions. First of all the MacDonalds functions K_p fulfill the recursion formula

$$\frac{d}{dt}[t^{-p}K_p(t)] = -t^{-p}K_{p+1}(t),$$

second these functions are linked with the Bessel functions of third order, the Hankel functions of second order $H_p^{(2)}$ in the following way:

$$K_p(t) = -\frac{1}{2}\pi i e^{-\frac{1}{2}ip\pi} H_p^{(2)}\left(te^{-\frac{1}{2}i\pi}\right) \qquad \left(-\frac{1}{2}\pi < \arg t < \pi\right).$$

For our singular integral operator is important the behavior $t \to 0$. For the Hankel functions are valid

$$H_p^{(2)}(t) \sim +i\left(\frac{2}{p}\right)^p \frac{\Gamma(p)}{\pi}, \ t \to 0, \qquad (p > 0).$$

Thus, we derive

$$\frac{d}{d(a_0|x|)}K_{a_0}(|x|) = \frac{1}{(2\pi)^{\frac{n}{2}}}a_0^{n-2} \cdot a_0 \frac{d}{d\,(a_0|x|)}\left(a_0|x|\right)^{-(\frac{n}{2}-1)} K_{\frac{n}{2}-1}(a_0|x|) =$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}}a_0^{n-1}(-1)\left(\frac{1}{a_0|x|}\right)^{\frac{n}{2}-1} K_{\frac{n}{2}}(a_0|x|).$$

Hence,

$$(D - ia_0)K_{a_0}(x) = \left(\sum_{j=1}^{n} \frac{x_j e_j}{|x|}\right) a_0 \frac{dK_{a_0}(|x|)}{d(a_0|x|)} - ia_0 K_{a_0}(|x|) =$$

$$= \frac{-1}{(2\pi)^{\frac{n}{2}}} (a_0|x|)^{\frac{n}{2}} K_{\frac{n}{2}}(a_0|x|) \sum_{j=1}^{n} \frac{x_j e_j}{|x|^n} - \frac{ia_0}{(2\pi)^{\frac{n}{2}}} \frac{1}{|x|^{n-2}} (a_0|x|)^{\frac{n}{2}-1} K_{\frac{n}{2}-1}(a_0|x|).$$

and we have

$$e_{ia}(x) = e^{-i<a,x>}\{(D -^{ia_0} M)K_{a_0}(x)\} = e^{-i<a,x>}\{(D - ia_0)K_{a_0}(x)\} =$$

$$= \frac{-e^{-i<a,x>}}{(2\pi)^{\frac{n}{2}}} \left\{ \sum_{j=1}^{n} \frac{x_j e_j}{|x|^n} (a_0|x|)^{\frac{n}{2}} K_{\frac{n}{2}}(a_0|x|) + \frac{ia_0}{|x|^{n-2}} (a_0|x|)^{\frac{n}{2}-1} K_{\frac{n}{2}-1}(a_0|x|) \right\}.$$

Using the properties of modified Bessel functions we get

$$\frac{e^{-i<a,x>}}{(2\pi)^{\frac{n}{2}}} (a_0|x|)^{\frac{n}{2}} K_{\frac{n}{2}}(a_0|x|) = \frac{e^{-i<a,x>}}{(2\pi)^{\frac{n}{2}}} (a_0|x|)^{\frac{n}{2}} \left(-\tfrac{1}{2}\right) \pi i (-i)^{\frac{n}{2}} H^{(2)}_{\frac{n}{2}}(-ia_0|x|) =$$

$$= \frac{e^{-i<a,x>}}{(2\pi)^{\frac{n}{2}}} (a_0|x|)^{\frac{n}{2}} \left(-\tfrac{1}{2}\right) \pi i (-i)^{\frac{n}{2}} i \left(\tfrac{2}{-ia_0|x|}\right)^{\frac{n}{2}} \frac{\Gamma\left(\frac{n}{2}\right)}{\pi} + \mathcal{O}(|x|^{\tau}) =$$

$$= \frac{1}{2\pi^{\frac{n}{2}}} \Gamma\left(\tfrac{n}{2}\right) + \mathcal{O}(|x|^{\tau}) = \frac{1}{\sigma_n} + \mathcal{O}(|x|^{\tau}), \ \tau > 0, \ \text{as } |x| \to 0.$$

We simply write $e(x)$ instead of $e_0(x)$, i.e.

$$e(x) = \frac{-1}{\sigma_n} \frac{x}{|x|^n}$$

and thus

$$e_{ia} - e(x) = \mathcal{O}(|x|^{-n+1+\tau}), \ \tau > 0,$$

where $e(x) = e_0(x)$ and also for every component, $j = 1, 2, \dots n$, we have

$$(e_{ia}(x))_j - (e(x))_j = \mathcal{O}(|x|^{-n+1+\tau}), \ \tau > 0.$$

Now, let

$$T_j u := \int_G (e_{ia}(x-y))_j u(y) dy, \ j = 1, 2, \dots, n,$$

then

$$T_j u = \int_G (e(x-y))_j u(y) dy + \int_G \{(e_{ia}(x-y))_j - (e(x-y))_j\} u(y) dy$$

and thus (cf. [16])

$$\frac{\partial T_j}{\partial x_k}u = \int_G \frac{\partial}{\partial x_k}(e(x-y))_j u(y)dy +$$

$$\int_G \frac{\partial}{\partial x_k}\left\{(e_{ia}(x-y))_j - (e(x-y))_j\right\} u(y)dy - (n-2)\int_{S^n}(x_j - y_j)\cos(r, x_k)dS^n$$

$$= \int_G \frac{\partial}{\partial x_k}(e_{ia}(x-y))_j u(y)dy - \delta_{jk}\frac{(n-2)}{n}\sigma_n \cdot u(x)$$

and the main part of the integral operator

$$\int_G \frac{\partial}{\partial x_k}(e(x-y))_j u(y)dy = -\frac{1}{\sigma_n}\int_G \frac{\partial}{\partial x_k}\left\{\frac{(x_j-y_j)}{|x-y|^n}\right\}u(y)dy$$

is a singular integral operator of Calderon-Zygmund-type. To sumrize, the operator $D_{ia}\mathcal{P}T_{ia}$ is a (general) singular integral operator.

5.2 Properties of the singular integral operator $D_{ia}\mathcal{P}T_{ia}$

Proposition 2 (cf. [16]) *If the symbol of a singular integral operator does not depend on the pole then that operator is bounded in $L_2(\mathbb{R}^m)$.*

Therefore, the operator $D_{ia}\mathcal{P}T_{ia}$ is bounded in $L_2(\mathbb{R}^m)$.

Theorem 2 *The operator $D_{ia}\mathcal{P}T_{ia}$ is monotone in $L_2(\mathbb{R}^m)$ and has a norm less than 1.*

Proof: Because $\mathcal{D}_{\mathcal{C}\ell}(\mathbb{R}^n)$ is dense in $L_{\mathcal{C}\ell,2}(G)$ we get with $v \in \mathcal{D}_{\mathcal{C}\ell}(\mathbb{R}^n)$

$$(D_{ia}\mathcal{P}T_{ia}v\,,\, v)_G = (D_{ia}\mathcal{P}T_{ia}v\,,\, D_{ia}T_{ia}v)_G$$

$$= ||D_{ia}\mathcal{P}T_{ia}v||^2_{2,G} + (D_{ia}\mathcal{P}T_{ia}v\,,\, D_{ia}\mathcal{Q}T_{ia}v)_G$$

and

$$(D_{ia}\mathcal{P}T_{ia}v\,,\, D_{ia}\mathcal{Q}T_{ia}v)_G =$$

$$= \mathcal{R}e\int_G D_{ia}\widetilde{\mathcal{P}T_{ia}v}\cdot D_{ia}\mathcal{Q}T_{ia}v dx = \mathcal{R}e\int_G -D_{i\bar{a}}\overline{\mathcal{P}T_{ia}v}\cdot D_{ia}\mathcal{Q}T_{ia}v dx.$$

On the other hand, we have

$$\mathcal{R}e\int_G -D\{\overline{\mathcal{P}T_{ia}v}\cdot D_{ia}\mathcal{Q}T_{ia}v\}dx =$$

$$= \mathcal{R}e\int_G -D\overline{\mathcal{P}T_{ia}v}\cdot D_{ia}\mathcal{Q}T_{ia}v dx + \mathcal{R}e\int_G \overline{\mathcal{P}T_{ia}v}\cdot(-D)D_{ia}\mathcal{Q}T_{ia}v dx =$$

$$= \mathcal{R}e \int_G -D\overline{\mathcal{P}T_{ia}v} \cdot D_{ia}\mathcal{Q}T_{ia}v dx + \mathcal{R}e \int_G \overline{\mathcal{P}T_{ia}v} \cdot \Delta\mathcal{Q}T_{ia}v dx +$$

$$-\mathcal{R}e \int_G \overline{\mathcal{P}T_{ia}v} \cdot D^{ia}M\mathcal{Q}T_{ia}v dx =$$

$$= \mathcal{R}e \int_G -D\overline{\mathcal{P}T_{ia}v} \cdot D_{ia}\mathcal{Q}T_{ia}v dx - \mathcal{R}e \int_G \overline{\mathcal{P}T_{ia}v} \cdot D^{ia}M\mathcal{Q}T_{ia}v dx$$

Hence, we obtain

$$(D_{ia}\mathcal{P}T_{ia}v\,,\, D_{ia}\mathcal{Q}T_{ia}v)_G =$$

$$\mathcal{R}e \int_G -D\{\overline{\mathcal{P}T_{ia}v} \cdot D_{ia}\mathcal{Q}T_{ia}v\} dx + \mathcal{R}e \int_G \overline{\mathcal{P}T_{ia}v} \cdot D^{ia}M\mathcal{Q}T_{ia}v dx +$$

$$+\mathcal{R}e \int_G {}^{-i\bar{a}}M\overline{\mathcal{P}T_{ia}v} \cdot (D +{}^{ia}M)\mathcal{Q}T_{ia}v\} dx =$$

$$= \mathcal{R}e \int_G -D\{\overline{\mathcal{P}T_{ia}v} \cdot D_{ia}\mathcal{Q}T_{ia}v\} dx + \mathcal{R}e \int_G \overline{\mathcal{P}T_{ia}v} \cdot D^{ia}M\mathcal{Q}T_{ia}v\} dx +$$

$$+\mathcal{R}e \int_G {}^{-i\bar{a}}M\overline{\mathcal{P}T_{ia}v} \cdot D\mathcal{Q}T_{ia}v\} dx + \mathcal{R}e \int_G \overline{\mathcal{P}T_{ia}v} \cdot ({}^{-i\bar{a}}M{}^{ia}M)\mathcal{Q}T_{ia}v dx. \qquad (1)$$

Due to $-i\bar{a}ia = \bar{a}a = |a|^2 \in \mathbb{R}$ we have ${}^{-i\bar{a}}M{}^{ia}M ={}^{|a|^2}M$ and thus

$$({}^{-i\bar{a}}M{}^{ia}M)\mathcal{Q}T_{ia}v ={}^{|a|^2}M\mathcal{Q}T_{ia}v = \mathcal{Q}{}^{|a|^2}MT_{ia}v \in \operatorname{im}\mathcal{Q}$$

and the last integral in (1) is equal to zero. Moreover, we have

$${}^{-\bar{a}}MD + D^{a}M ={}^{\mathcal{Q}a}MD + D^{\mathcal{Q}a}M = 2\mathcal{P}({}^{a}MD)$$

and thus

$$\mathcal{R}e \int_G \overline{\mathcal{P}T_{ia}v} \cdot D^{ia}M\mathcal{Q}T_{ia}v\} dx + \mathcal{R}e \int_G {}^{-i\bar{a}}M\overline{\mathcal{P}T_{ia}v} \cdot D\mathcal{Q}T_{ia}v\} dx =$$

$$= \mathcal{R}e \int_G \overline{\mathcal{P}T_{ia}v} \cdot{}^{-\bar{a}}MD + D^{a}M\mathcal{Q}T_{ia}v\} dx = \mathcal{R}e \int_G \overline{\mathcal{P}T_{ia}v} \cdot 2\mathcal{Q}\mathcal{P}^{a}MDT_{ia}v\} dx = 0.$$

To sum up, we obtain

$$(D_{ia}\mathcal{P}T_{ia}v\,,\, v)_G = \mathcal{R}e \int_G -D\left\{\overline{\mathcal{P}T_{ia}v} \cdot D_{ia}\mathcal{Q}T_{ia}v\right\} dx.$$

Using Gauß's formula we get

$$(D_{ia}\mathcal{P}T_{ia}v\,,\, v)_G = \mathcal{R}e \int_\Gamma -n \cdot \left\{\overline{\mathcal{P}T_{ia}v} \cdot D_{ia}\mathcal{Q}T_{ia}v\right\} dx.$$

In an analogously way we derive

$$(D_{ia}\mathcal{P}T_{ia}v\,,\,v)_{\mathbf{R}^n\setminus\bar{G}} = \mathcal{R}e\int\limits_{\Gamma} n\cdot\left\{\overline{\mathcal{P}T_{ia}v}\cdot D_{ia}\mathcal{Q}T_{ia}v\right\}dx.$$

Due to Borel-Pompeiu formula we have

$$0 = ||D_{ia}T_{ia}v||^2_{\mathbf{R}^n\setminus\bar{G}} =$$

$$||D_{ia}\mathcal{P}T_{ia}v||^2_{\mathbf{R}^n\setminus\bar{G}} + 2(D_{ia}\mathcal{P}T_{ia}v, D_{ia}\mathcal{Q}T_{ia}v)_{\mathbf{R}^n\setminus\bar{G}} + ||D_{ia}\mathcal{Q}T_{ia}v||^2_{\mathbf{R}^n\setminus\bar{G}}.$$

and thus

$$\begin{aligned}(D_{ia}\mathcal{P}T_{ia}v, D_{ia}\mathcal{Q}T_{ia}v)_G &= -(D_{ia}\mathcal{P}T_{ia}v, D_{ia}\mathcal{Q}T_{ia}v)_{\mathbf{R}^n\setminus\bar{G}}\\ &= \tfrac{1}{2}\left\{||D_{ia}\mathcal{P}T_{ia}v||^2_{\mathbf{R}^n\setminus\bar{G}} + ||D_{ia}\mathcal{Q}T_{ia}v||^2_{\mathbf{R}^n\setminus\bar{G}}\right\}\end{aligned}$$

This leads to the estimation

$$(D_{ia}\mathcal{P}T_{ia}v, v)_G = ||D_{ia}\mathcal{P}T_{ia}v||^2_G + \frac{1}{2}\left\{||D_{ia}\mathcal{P}T_{ia}v||^2_{\mathbf{R}^n\setminus\bar{G}} + ||D_{ia}\mathcal{Q}T_{ia}v||^2_{\mathbf{R}^n\setminus\bar{G}}\right\} \geq 0$$

which proves the positivity of the operator $D_{ia}\mathcal{P}T_{ia}$. Moreover, we get

$$||v||^2_G = (v, v)_G =$$

$$= ||D_{ia}\mathcal{P}T_{ia}v||^2_G + 2(D_{ia}\mathcal{P}T_{ia}v, D_{ia}\mathcal{Q}T_{ia}v)_G + ||D_{ia}\mathcal{Q}T_{ia}v||^2_G \geq ||D_{ia}\mathcal{P}T_{ia}v||^2_G$$

i.e. the norm of the operator $D_{ia}\mathcal{P}T_{ia}$ is not greater than 1.

5.3 Nonlinear integral equations

Let us denote by $PVecL_{2,\mathcal{C}\ell}$ the subspace of all paravectors, i.e. elements of the form $u = \sum\limits_{j=0}^{n} u_j e_j$, of $L_{2,\mathcal{C}\ell}$. It is easy to see that $D_{ia}\mathcal{P}T_{ia}$ maps $PVecL_{2,\mathcal{C}\ell}$ into $PVecL_{2,\mathcal{C}\ell}$ and is a monotone operator. If additional $a_0 = 0$ then $D_{ia}\mathcal{P}T_{ia}$ maps $VecL_{2,\mathcal{C}\ell}$, i.e. the subspace of all vectors $u = \sum\limits_{j=1}^{n} u_j e_j$ into itself. Therefore, we can use monotonicity principles to get information about the following problem:

$$\begin{aligned} Au = g, &\qquad g \in PVecL_{2,\mathcal{C}\ell},\\ \text{where}\quad Au = F(x,u) + Bu, &\quad Bu = D_{ia}\mathcal{P}T_{ia}u.\end{aligned}$$

6 Some remarks

6.1 Representation formulae

The operator $D_{ia}\mathcal{P}T_{ia}$ may by represented as

$$D_{ia}\mathcal{P}T_{ia}u = D_{ia}\mathcal{P}\left(\int_G \left(e^{-i<a,x-y>}D_{-ia_0}K_{ia_0}(x-y)\right)u(y)dy\right)$$

In the case $a = 0$ we get simply

$$D\mathcal{P}Tu = \frac{1}{2^n\pi^{n/2}}D\mathcal{P}D\left(\int_G \frac{1}{\sigma_n}\left(\frac{1}{|x-y|^{n-2}}\right)u(y)dy\right)$$

6.2 An application

In case of $n = 3$ the operator $D\mathcal{P}T$ plays an important role in magnetic field calculations ([5]). The general problem is to solve the nonlinear singular integral equation

$$F(M(x), x) - \frac{1}{4\pi}\,\text{grad div}\int_G \frac{M(y)}{|x-y|}dy = H_a(x)$$

for the magnatization $M(x) = (M_1(x), M_2(x), M_3(x))$ given an "applied field" $H_a(x)$. G is a bounded region in $\mathbb{R}^3$, which is imagined to be filled with a ferromagnetic material. $H(x) = F(M(x), x)$ is the net field in G considered as a function of M; F is, as a rule, an experimental function representing the magnetic permeability of the of the ferromagnetic material, which varies with the magnetization. this relationship is ussally given by a single valued magnetization curve, called the $M - H$ characteristic of the magnetic material; obtained by neglecting the hysteresis effects. The integral represents the demagnetization field due to spatial distribution of magnetization.

Let us denote by $H_i(x)$ the field $H(x) - H_a(x)$ induced by $M(x)$. Then the monotonicity (=positivity) of the operator $D\mathcal{P}T$ leads to

$$(H_i, M) \leq 0 \tag{2}$$

and norm not greater than one gives

$$-(H_i, M) \leq ||M||^2 \text{ or } \left|\left(H_i, \frac{M}{||M||}\right)\right| \leq ||M||. \tag{3}$$

The relation 2 indicates that the mean angle between H_i and M in G is not less than $\frac{\pi}{2}$ and 3 that the mean value of the projection of H_i onto M is not more than the mean value of M in G.

This coincides with the well-known maxims of electrical engineers that "the induced field is directed opposite of the net field, or magnatization", and "the induced field is less than the magnatization."

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