Nonlinear singular integral equations involving the Hilbert transformation in Clifford Analysis

by

Swanhild Bernstein

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Swanhild Bernstein[1]

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Dedicated to Prof. Dr. L. von Wolfersdorf on the occasion of his 65th birthday

Abstract. We apply operator theoretical methods for monotone and maximal monotone operators to prove the existence of solutions for nonlinaer singular integral and integro-differential equations involving the Hilbert transformation in the Clifford algebra $\mathcal{C}\ell_{n,0}$. The properties of the Hilbert transform are proved using Clifford analysis. We generalize well-known results concerning the complex Hilbert transformation and the singular Cauchy integral operator to higher dimensions.

Keywords *Clifford analysis, Cauchy-type integrals, nonlinear integro-differential equations*

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1 Introduction

Clifford analysis is a generalisation of complex function theory to higher dimensions. Therefore it is natural to look for similarities. Methods of monotone operator theory as given in [6] and [7] have been applied to nonlinear singular integral equations on the unit circle and on the real line. The basic property used here is the monotonicity = positivity of the Hilbert transform. To get an overview of these results we recommend the papers [15] and [16].
A generalisation from the real line to the complex plane is done in [1].

Singular integral operators, especially the Cauchy transform, play an important role in Quaternionic and Clifford analysis. We want to give some outline about Clifford analysis and singular integral operators.
A foundation of Clifford analysis was done in [5], Quaternionic analysis is treated extensivly in [9] and more recently in [10] and [11]. These last books explain also some relations to physical problems. Connections between harmonic and monogenic functions are discussed in [8]. The Cauchy transform and some classes of singular integral operators and associated equations in a quaternionic context were investigated in [14] concerning Fredholm property. Cauchy transform and convolution singular integral operators on Lipschitz surfaces using Clifford analytical techniques were treated in [12] and [13].
A good overview of monotonicity principles and their application to operator equations is given in [17]. Special nonlinear singular integral equations were considered in [3] and [4].

Our paper is organized as follows. First we review some basic properties of Clifford algebras, Clifford function theory and related function spaces. Then we prove mapping and monotonicity properties of the Hilbert transform and related operators, especially the Nemyckii operator. In the final section we apply monotone operator theory to several kind of nonlinear singular integral equations involving the Hilbert transform.

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2 Preliminaries

We shall briefly review some basic definitions and properties of the function theory corresponding to the Clifford algebra. For a more detailed investigation of this matters, we refer to [5], [8], [9], [10] and [11].

Let $\{e_1, e_2, \ldots, e_m\}$ be an orthonormal basis in $\mathbb{R}^m$. Consider the 2^m-dimensional real Clifford algebra $\mathcal{C}\ell_{m,0}$ generated by $e_1, \ldots, e_m$ according to the multiplication rules

$$e_i e_j + e_j e_i = 2\delta_{ij} e_0,$$

where e_0 is the identity of $\mathcal{C}\ell_{m,0}$. The elements $e_I : I = \{h_1, \ldots, h_k\} \subseteq \{1, \ldots, m\}$ define a basis of $\mathcal{C}\ell_{m,0}$, where $e_I = e_{h_1} \cdots e_{h_k}$, $1 \leq h_1 < \ldots h_k \leq m$, and $e_\emptyset = e_0$. thus, an arbitrary element a of $\mathcal{C}\ell_{m,0}$ may be represented as

$$a = \sum_I a_I e_I, \ a_I \in \mathbb{R}.$$

Especially the elements $\vec{x} \in \mathbb{R}^m$ will be identified with $\sum_{j=1}^m x_j e_j \in \mathcal{C}\ell_{m,0}$. We want to denote be $Sc\, a = a_0 e_0 = a_0$ the *scalar part* of a and by $Vec\, a = a - Sc\, a$ the *(multi-) vector part.*
We introduce an automrphism called *reversion.* The reversed element $\hat{a}$ of a is given by $\hat{a} := \sum_I b_I \hat{e}_I$, where $\hat{e}_0 = e_0$, $\hat{e}_j = e_j$, $\hat{e}_I = \hat{e}_{h_k} \cdots \hat{e}_{h_2} \cdot \hat{e}_{h_1}$. Then by

$$[a, b] := Sc\, \hat{a}b = Sc\, a\hat{b} = \sum_I a_I b_I \text{ and } |a|^2 = Sc\, \hat{a}a = Sc\, a\hat{a}$$

are given a scalar product and a norm in $\mathcal{C}\ell_{m,0}$ and we have especially $x\hat{x} = \sum_{j=1}^m x_j^2 = |x|^2$.

We suppose $G \subset \mathbb{R}^m$ to be a domain with a smooth boundary Γ. We consider functions $f(x)$ definded on G with values in $\mathcal{C}\ell_{m,0}$. These functions may be written as

$$f(x) = \sum_I f_I(x) e_I, \quad x \in G.$$

Properties such as continuity, differentiability, integrability, and so on, which are ascribed to f have to be possessed by all components $f_I(x)$. In this way the usual Banach spaces of these functions are denoted by C^α, L^2, H^1 and H_0^1. $H^1(G) = \{u \in L^2(G) : \frac{\partial u}{\partial x_k} \in L^2(G)\}$ and $H_0^1(G)$ is the closure of $C_0^\infty(G)$ in H^1. We now define the Dirac operator by

$$D = \sum_{k=1}^m e_k \frac{\partial}{\partial x_k}.$$

We consider in G the equation

$$(Du)(x) = 0,$$

and look for its solutions which are called *left-monogenic* functions in G.
Now we define the Cauchy kernel in $\mathbb{R}^m$ by

$$e(x) = \frac{1}{\sigma_m} \frac{x}{|x|^m}, \ x \neq 0, \ \sigma_m = \frac{2\pi^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}\right)}.$$

It is well known that $e(x)$ (a fundamental solution of D) is monogenic in $\mathbb{R}^m \setminus \{0\}$. Using the function $e(x)$ we introduce the following integral operators:

$$(T_G u)(x) := \int_G e(x-y)u(y)dy, \quad x \in \mathbb{R}^m \qquad \text{(Teodorescu transform)}$$

$$(F_\Gamma u)(x) := -\int_\Gamma e(x-y)n(y)u(y)d\Gamma_y, \quad x \notin \Gamma \qquad \text{(Cauchy type operator)}$$

$$(S_\Gamma u)(x) := -\int_\Gamma 2e(x-y)n(y)u(y)d\Gamma_y, \ x \in \Gamma \qquad \text{(singular integral operator)}$$

where $n(y) = \sum_{i=1}^{m} e_i n_i(y)$ is the outward pointing normal (unit) vector to Γ at the point y. The integral which defines the operator S_Γ has to be taken in the sense of Cauchy's principle value. We remark that the operators $F_\Gamma, S_\Gamma, P_\Gamma$, and Q_Γ are defined in spaces of Hölder continuous functions. It is possible to extend these operators to Sobolev spaces in the classical way by approximation (with Hölder continuous functions). We omit the detailed discussion here.
We introduce weighted L^2-spaces. Let G be a bounded or unbounded smooth domain in $\mathbb{R}^m$

$$L^{2,\alpha}(G, \mathcal{C}\ell_{m,0}) := \{u : (1+|x|^2)^{\frac{\alpha}{2}} u \in L^2(G, \mathcal{C}\ell_{m,0})\}.$$

These spaces are (real) Hilbert space with the scalar product

$$(u,v)_\alpha = Sc \int_G (1+|x|^2)^\alpha \hat{u}(x)v(x)\, dx = \int_G (1+|x|^2)^\alpha [u(x), v(x)]\, dx,$$

and the norm is $||u||_\alpha^2 = (u,u)_\alpha$. We set $(.,.) = (.,.)_0$ and $||.|| = ||.||_0$. Further we will use the weighted Sobolv-spaces

$$H^{1,\alpha}(G, \mathcal{C}\ell_{m,0}) = \{(1+|x|^2)^{\frac{\alpha}{2}} u \in L^2(G, \mathcal{C}\ell_{m,0}),\ (1+|x|^2)^{\frac{\alpha+1}{2}} Du \in L^2(G, \mathcal{C}\ell_{m,0})\}$$

It is easy to see that if G is a bounded domain these weighted spaces coincides with $L^2(G, \mathcal{C}\ell_{m,0})$ and $H^1(G, \mathcal{C}\ell_{m,0})$ respectivly. From [2] we immediately get the following statements.

Lemma 1 *Let* $u \in H^{1,\alpha-1}(G, \mathcal{C}\ell_{m,0})$, $-\frac{m}{2}+1 < \alpha < \frac{m}{2}$. *Then we have*

$$\text{(i)} \qquad F_\Gamma u + T_G Du = \begin{cases} u & , \quad x \in G \\ 0 & , \quad x \in \mathbb{R}^m \setminus \overline{G} \end{cases} \qquad \textit{(Borel–Pompeiu's formula)}$$

$$\text{(ii)} \qquad DT_G u = \begin{cases} u & , \quad in\ G \\ 0 & , \quad in\ \mathbb{R}^m \setminus \overline{G} \end{cases}$$

$$\text{(iii)} \qquad DF_\Gamma u = 0 \ \ in\ G \cup (\mathbb{R}^m \setminus \overline{G}).$$

Lemma 2 (Plemelj–Sokhotzkij's formulas) *Let* $u \in C^{0,\alpha}(G, \mathcal{C}\ell_{m,0})$, $0 < \alpha < 1$. *Then we have*

$$\text{(i)} \quad \lim_{\substack{x\to\xi\in\Gamma \\ x\in G}} (F_\Gamma u)(x) = P_\Gamma u(\xi), \qquad \text{(ii)} \quad \lim_{\substack{x\to\xi\in\Gamma \\ x\in\mathbb{R}^m\setminus\overline{G}}} (F_\Gamma u)(x) = -Q_\Gamma u(\xi)\,.$$

for any $\xi \in \Gamma$.

The operator $P_\Gamma := \frac{1}{2}(I + S_\Gamma)$ denotes the projection onto the space of all $\mathcal{C}\ell_{m,0}$-valued functions which have a left monogenic extension into the domain G.

$Q_\Gamma := \frac{1}{2}(I - S_\Gamma)$ denotes the projection onto the space of all $\mathcal{C}\ell_{m,0}$-valued functions which have a left monogenic extension into the domain $\mathbb{R}^m \setminus \overline{G}$ and vanish at infinity.

Corollary 1 *Let $u \in L^2(\Gamma, \mathcal{C}\ell_{m,0})$. Then the equations*

(i) $S_\Gamma^2 u = u$ *(ii)* $F_\Gamma P_\Gamma u = F_\Gamma u$

(iii) ${P_\Gamma}^2 u = P_\Gamma u$ *(iv)* ${Q_\Gamma}^2 u = Q_\Gamma u$

are valid on Γ.

Corollary 2 *Let $u \in H^{1,\alpha-1}(G, \mathcal{C}\ell_{m,0})$, $-\frac{m}{2} + 1 < \alpha < \frac{m}{2}$. Then*

$$T_G D u = u \text{ iff } tr\, u \in \operatorname{im} Q_\Gamma \text{ and } TDu = u \text{ in } \mathbb{R}^m.$$

3 Monogenicity and Hilbert transformation

Lemma 3 *Let $W \in H^{1,\alpha}(\mathbb{R}^{n+1}_+, \mathcal{C}\ell_{n+1,0})$ be a monogenic function then there exist $U, V \in H^{1,\alpha}(\mathbb{R}^{n+1}_+, \mathcal{C}\ell_{n,0})$ such that $W = U + e_{n+1}V$ and $\Delta U = \Delta V = 0$.*

Proof: Because of $\mathcal{C}\ell_{n+1,0} = \mathcal{C}\ell_{n,0} + e_{n+1}\mathcal{C}\ell_{n,0}$ there exist $U, V \in \mathcal{C}\ell_{n,0}$ such that $W = U + e_{n+1}V$. We want to denote be D the Dirac operator in $\mathbb{R}^n$, i.e. $D = \sum\limits_{k=1}^{n} e_k \frac{\partial}{\partial x_k}$. Then

$$\left(D + e_{n+1}\frac{\partial}{\partial x_{n+1}}\right) W = \left(D + e_{n+1}\frac{\partial}{\partial x_{n+1}}\right)(U + e_{n+1}V) = 0$$

$$\Longleftrightarrow \quad DU - e_{n+1}DV + e_{n+1}\frac{\partial}{\partial x_{n+1}}U + \frac{\partial}{\partial x_{n+1}}V = 0$$

$$\Longleftrightarrow \quad \begin{cases} DU + \frac{\partial}{\partial x_{n+1}}V = 0 \\ \frac{\partial}{\partial x_{n+1}}U - DV = 0 \end{cases} \tag{1}$$

Thus $\Delta U = \Delta V = 0$ in $\mathbb{R}^{n+1}_+$, where Δ is the Laplacian in $\mathbb{R}^{n+1}$.•

Let G be a bounded or unbounded smooth domain in $\mathbb{R}^n$ then we define the *Hilbert transformation* by

$$(H_G u)(x) = \int_G 2e(x-y)u(y)dy.$$

If G is the hole space $\mathbb{R}^n$ we denote the Hilbert transformation by H. If we interprete $G \subset \mathbb{R}^n$ as a subset of the boundary $\mathbb{R}^n$ of $\mathbb{R}^{n+1}_+$ with outer normal $-e_{n+1}$ we get $H_G(-e_{n+1})u = S_G u$.

Properties of H_G, H and HD

Theorem 1 *Let $u, v \in L^2(G, \mathcal{C}\ell_{n,0})$ then*

(i) $H_G : L^{2,\alpha}(G, \mathcal{C}\ell_{n,0}) \to L^{2,\alpha}(G, \mathcal{C}\ell_{n,0})$,
$H : L^{2,\alpha}(\mathbb{R}^n, \mathcal{C}\ell_{n,0}) \to L^{2,\alpha}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$, $-\frac{n}{2} < \alpha < \frac{n}{2}$.

(ii) $(H_G u, v) = -(u, H_G v)$, $(Hu, v) = -(u, Hv)$, $u \in L^{2,\alpha}(G, \mathcal{C}\ell_{n,0})$ *and* $v \in L^{2,-\alpha}(G, \mathcal{C}\ell_{n,0})$, $\frac{-n}{2} < \alpha < \frac{n}{2}$,

(iii) $(H_G u, u) = 0$, $(Hu, u) = 0$, $u \in L^{2,\alpha}(G, \mathcal{C}\ell_{n,0})$, $0 \le \alpha < \frac{n}{2}$,

(iv) $H^2 = -I$, $u \in L^{2,\alpha}(G, \mathcal{C}\ell_{n,0})$, $-\frac{n}{2} < \alpha < \frac{n}{2}$.

Proof: The property (i) follows from the fact that H is a singular integral operator. To prove (ii) let $x \in G$ be fixed and $y \in G$ variable and put

$$u_N(y) = \begin{cases} u(y), & \text{if } |y-x| \geq \frac{1}{N}, \\ 0, & \text{if } |y-x| < \frac{1}{N}. \end{cases}$$

Then $||u_N - u|| \to 0$ and consequently $||w_N - w|| \to 0$, where $w_N(x) = \int_G \hat{u_N}(y)e(\widehat{x-y})dy$ and $w(x) = \int_G \hat{u}(y)e(\widehat{x-y})dy$. This and Hölders inequality leads to

$$\int_G w(x)v(x)\,dx = \lim_{N\to\infty}\int_G w_N(x)v(x)\,dx = \lim_{N\to\infty}\int_G\int_G \hat{u}(y)e(\widehat{x-y})\,dy\,v(x)\,dx.$$

In the last integral the order of integration can be reversed and because of $\hat{u_N}(y)e(\widehat{x-y})v(x) = \hat{u}(y)e(\widehat{x-y})v_N(x)$ we get

$$\int_G w(x)v(x)\,dx = \lim_{N\to\infty}\int_G\int_G \hat{u}(y)e(\widehat{x-y})\,dy\,v_N(x)\,dx = \int_G \hat{u}(y)\int_G -e(y-x)\,v(x)\,dx\,dy.$$

The property (iii) follows from

$$(H_G u, u) = -(u, H_G u) = -(H_G u, u).$$

For the second relation we use the embeding $L^{2,\alpha}(\mathbb{R}^n, \mathcal{C}\ell_{n,0}) \subset L^2(\mathbb{R}^n, \mathcal{C}\ell_{n,0}) \subset L^{2,-\alpha}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$ Thus H maps $L^{2,\alpha}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$ into $L^{2,-\alpha}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$, $\frac{n}{2} > \alpha \geq 0$. To prove (iv) we remember that $S^2 = I$ and thus

$$H^2 u = -H(-e_{n+1})H(-e_{n+1})u = -S^2 u = -u. \bullet$$

Lemma 4 *Let* $u \in L^{2,\alpha}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$, $-\frac{n}{2} < \alpha\frac{n}{2}$, *then there exists a* $v \in L^{2,\alpha}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$ *such that* $w = u + e_{n+1}v \in \operatorname{im} P_{\mathbb{R}^n}$.

Proof: Assume $w = u + e_{n+1}v \in \operatorname{im} P_{\mathbb{R}^n}$. Then

$$\begin{aligned} Sw = w \text{ on } \mathbb{R}^n \quad &\Longleftrightarrow \quad H(-e_{n+1})w = H(-e_{n+1})(u + e_{n+1}v) = w = u + e_{n+1}v \\ &\Longleftrightarrow \quad H(-v - e_{n+1}u) = e_{n+1}Hu - Hv = u + e_{n+1}v \\ &\Longleftrightarrow \quad \begin{cases} -Hv = u \\ Hu = v \end{cases} \end{aligned} \tag{2}$$

Now, set $v := Hu$ then $Hv = H^2 u = -u$ and going backwards inside the relations given before we obtain the desired relation. •

Theorem 2 *Let* $u, v \in H^{1,-\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$ *then*

$$(HDu, v) = (u, DHv) \text{ and } (-HDu, u) \geq 0.$$

Proof: First off all we have

$$(Du, v) = Sc\sum_{k=1}^{n}\int_{\mathbb{R}^n} e_k \widehat{\frac{\partial}{\partial x_k}u(x)}\, v(x)dx =$$

$$= Sc\sum_{k=1}^{n}\int_{\mathbb{R}^n} \frac{\partial}{\partial x_k}\hat{u}(x)\,\hat{e}_k\,v(x)\,dx = -Sc\sum_{k=1}^{n}\int_{\mathbb{R}^n} \hat{u}(x)e_k\frac{\partial}{\partial x_k}v(x)\,dx = -(u, Dv)$$

Put this together with (i) of Theorem 1 we get $(HDu, v) = -(Du, Hv) = (u, DHv)$.
Now, let be $u \in H^{1,-\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$ then using Lemma 3 and Lemma 4 we have with $v := Hu \in H^{1,-\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$ that $w = u + e_{n+1}v = trW = tr(U + e_{n+1}V)$, where W is a monogenic function in $\mathbb{R}^{n+1}_+$. Therefore if $U, V \in C^2(\overline{\mathbb{R}^{n+1}_+}, \mathcal{C}\ell_{n,0}(\mathbb{R}))$ we conclude from (1) and (2)

$$\frac{\partial}{\partial x_{n+1}} U|_{x_{n+1}=0} = tr\frac{\partial}{\partial x_{n+1}} U = trDV = DV|_{x_{n+1}=0} = DtrV = Dv = DHu$$

and thus

$$(-HDu, u) = (u, -DHu) = -(U|_{x_{n+1}=0}, \frac{\partial}{\partial x_{n+1}} U|_{x_{n+1}=0}) =$$

$$= -\sum_I \int_{\mathbb{R}^n} U_I|_{x_{n+1}=0} \frac{\partial}{\partial x_{n+1}} U_I|_{x_{n+1}=0} = \sum_I \int_{\mathbb{R}^{n+1}_+} U_I \Delta U_I \, dx + \sum_{k=1}^{n+1} \sum_I \int_{\mathbb{R}^{n+1}_+} \left(\frac{\partial}{\partial x_k} U_I\right)^2 dx \geq 0,$$

because the first integral is zero due to $\Delta U = 0$. The space $C^2(\overline{\mathbb{R}^n}, \mathcal{C}\ell_{n,0})$ is dense in $H^{1,-\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$ and we get the desired relation. •

The Nemyckii operator

We want to study two types of nonsingular integral equations. First, we require the properties of the so-called Nemyckii operator F in a Clifford analysis context. This operator is defined as

$$(Fu)(x) = f(x, u_0(x), u_1(x), \ldots, u_N(x)) = f(x, u(x)), \ N = 2^n$$

with $u = \sum_I u_I(x) e_I$. We make the following assumptions:
(A1) Carathéodory condition: Let $f : G \times \mathcal{C}\ell_{n,0} \to \mathcal{C}\ell_{n,0}$ be a given function, where G is a nonempty set in $\mathbb{R}^n$, $n, N \geq 1$. Moreover,

$$x \to f(x, u) \text{ is measurable on } G \text{ for all } u \in \mathcal{C}\ell_{n,0};$$

$$u \to f(x, u) \text{ is continuous on } \mathcal{C}\ell_{n,0} \text{ for almost all } x \in G.$$

(Aα) Growth condition: For all $(x, u) \in G \times \mathcal{C}\ell_{n,0}$, $\alpha \geq 0$,

$$(1 + |x|^2)^{-\frac{\alpha}{2}} |f(x, u)| \leq a(x) + b|u|(1 + |x|^2)^{\frac{\alpha}{2}}.$$

Here, b is a fixed positive number and $a \in L^{2,\alpha}(G)$ is a real-valued nonnegative function.

Proposition 1 *Under the two assumptions (A1) and (Aα), the following are valid:*
The Nemyckii operator

$$F : L^{2,-\alpha}(G, \mathcal{C}\ell_{n,0}) \to L^{2,\alpha}(G, \mathcal{C}\ell_{n,0})$$

is continuous and bounded with

$$||Fu||_{L^{2,\alpha}} \leq const(||a||_{L^{2,-\alpha}} + ||u||_{L^{2,-\alpha}})$$

and

$$(Fu, u) = Sc \int_G f(\widehat{x, u(x)}) u(x) dx \qquad \textit{for all } u \in L^{2,-\alpha}(G, \mathcal{C}\ell_{n,0}).$$

Moreover,

Monotonicity of f: The function f is monotone with respect to u i.e.

$$[f(x,u)-f(x,v),u-v]\geq 0$$

for all $u,v\in L^{2,-\alpha}(G,\mathcal{C}\ell_{n,0})$ implies F *is monotone.*
Strictly monotonicity of f: The function f is strictly monotone with respect to u i.e.

$$[f(x,u)-f(x,v),u-v]>0$$

for all $u,v\in L^{2,-\alpha}(G,\mathcal{C}\ell_{n,0})$ implies F *is strictly monotone.*
Coercivness of f:

$$[f(x,u),u]\geq d(1+|x|^2)^{-\alpha}|u|^2+g(x),$$

where $g\in L^1(G)$ implies F *is coercive* and

$$(Fu,u)\geq d||u||_{-\alpha}^2+\int\limits_G g(x)dx$$

for all $u\in L^{2,-\alpha}(G,\mathcal{C}\ell_{n,0})$.

4 Monotonicity principles for integral operators

In our considerations we will use the following theorem on maximal monotone operators by Browder [7].

Theorem 3 *Let X be a real (separable) reflexive Banach space and $A=A_1+A_2$, where $0\in D(A_1), A_1: D(A_1)\subset X\to X^*$ maximal monotone, and $A_2: X\to X^*$ bounded, monotone, coercive and (hemi-)continuous*
then A is surjective.

If A is strictly monotone then A is injective.

This theorem also holds if $A=A_2$. In our setting $X=X^*=L^2(G,\mathcal{C}\ell_{n,0})$ or $X=L^{2,-\frac{\alpha}{2}}(G,\mathcal{C}\ell_{n,0})$ and $X^*=L^{2,\frac{\alpha}{2}}(G,\mathcal{C}\ell_{n,0})$.

Integral equations

Theorem 4 (Hammerstein-type equations) *Let G be a bounded or unbounded smooth domain and $\phi(x,u)$ be a monotone, coercive Carathèodory function on $G\times\mathcal{C}\ell_{n,0}$ satisfying $(A\alpha)$ with $\alpha=0$ and let K be a linear bounded, positive operator from $L^2(G,\mathcal{C}\ell_{n,0})\to L^2(G,\mathcal{C}\ell_{n,0})$. Then*

$$u+(\lambda H_G+K)\phi u=f$$

has a solution $u\in L^2(G,\mathcal{C}\ell_{n,0})$ for any $f\in L^2(G,\mathcal{C}\ell_{n,0})$ and each $\lambda\in\mathbb{R}$. If Φ or K are strictly monotone this solution is unique.

Proof: This follows from [17] Theorem 32.B. •

Theorem 5 *Let G be a bounded or unbounded smooth domain and $\Phi(x,u)$ be a monotone, coercive Carathèodory function on $G\times\mathcal{C}\ell_{n,0}$ and K a linear bounded, positive operator from $L^{2,\alpha}(G,\mathcal{C}\ell_{n,0})\to L^{2,-\alpha}(G,\mathcal{C}\ell_{n,0})$, $\frac{n}{2}>\alpha\geq 0$. Then*

$$\Phi u+\lambda H_G u+Ku=g$$

has a solution $u\in L^{2,\alpha}(G,\mathcal{C}\ell_{n,0})$, $\frac{n}{2}>\alpha\geq 0$, for any $g\in L^{2,-\alpha}(G,\mathcal{C}\ell_{n,0})$, $\frac{n}{2}>\alpha\geq 0$, and each fixed $\lambda\in\mathbb{R}^n$. This solution is unique if $\Phi+\lambda H+K$ is strictly monotone.

Proof: Theorem 3 or for example [17] Theorem 26.A. •

Maximal monotone operators and integro-differential equations

For the consideration of integro-differential equations we will use the property of maximal monotonicity.

Definition 1 *([7]) A subset M of $X \times X^*$ is said to be a monotone set if for each pair $[u_1, w_1]$ and $[u_2, w_2]$ in M, we have*

$$(w_2 - w_1, u_2 - u_1) \geq 0.$$

Such a set M is said to be maximal monotone if it is maximal among monotone sets in the sense of inclusion, and a mapping A is said to be maximal monotone if its graph is a maximal monotone set.

Theorem 6 *Let G be a smooth domain in $\mathbb{R}^n$ and $\gamma(x)$ a real-valued, continuous, positiv function and there exist constants $C, c > 0$ such that $0 < \inf_{x\in G}(1+|x|^2)^{\frac{1}{2}}\gamma(x) \leq \sup_{x\in G}(1+|x|^2)^{\frac{1}{2}}\gamma(x) \leq C$. Then: the operator $A = D + \gamma(x)I$ defined on $D(A) = \{u \in H^{1,-\frac{1}{2}}(G, \mathcal{C}\ell_{n,0}),\ tr u \in \operatorname{im} Q_\Gamma\}$ is a maximal monotone mapping $D(A) \to L^{2,\frac{1}{2}}(G, \mathcal{C}\ell_{n,0})$.*

If $G = \mathbb{R}^n$ then $D(A) = H^{1,-\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{0,n})$.

Proof: The operator $\gamma(x)I$ maps $L^{2,-\frac{1}{2}}(G, \mathcal{C}\ell_{n,0})$ uniquely onto $L^{2,\frac{1}{2}}(G, \mathcal{C}\ell_{n,0})$ because of

$$||\gamma(x)u||^2_{\frac{1}{2}} = \int_G (1+|x|^2)^{\frac{1}{2}}|\gamma(x)|^2|u(x)|^2 dx \leq$$

$$\leq \sup_G\{(1+|x|^2)|\gamma(x)|^2\} \int_G (1+|x|^2)^{-\frac{1}{2}}|u(x)|^2 dx \leq C||u||^2_{-\frac{1}{2}}.$$

From Theorem 2 we get that $(Du, u) = 0$ hence

$$(Du + \gamma(x)u, u) = (Du, u) + (\gamma(x)u, u) = (\gamma(x)u, u) = \int_G \gamma(x)|u(x)|^2 dx \geq 0.$$

To prove maximal monotonicity we show the existence of a uniquely determined inverse operator with domain $L^{2,-\frac{1}{2}}(G, \mathcal{C}\ell_{n,0})$. From [2] we know that T_G maps $L^{2,\frac{1}{2}}(G, \mathcal{C}\ell_{n,0})$ into $L^{2,-\frac{1}{2}}(G, \mathcal{C}\ell_{n,0})$. It is easely seen that $(T_G u, u) = 0$. Therefore, the operator:

$$\gamma^{-1}(x)I + T_G : L^{2,\frac{1}{2}}(G, \mathcal{C}\ell_{n,0}) \to L^{2,-\frac{1}{2}}(G, \mathcal{C}\ell_{n,0})$$

is linear, bounded, strictly monotone and coercive due to

$$(\gamma^{-1}(x)v + T_G v, v) = (\gamma^{-1}(x)v, v) = \int_G \gamma^{-1}(x)(1+|x|^2)^{-\frac{1}{2}}(1+|x|^2)^{\frac{1}{2}}|v(x)|^2 dx \geq k||v||^2_{\frac{1}{2}}.$$

Now, because of $tr u \in \operatorname{im} Q_\Gamma$ we have

$$Du + \gamma(x)u = f \iff u + T_G\gamma(x)u = T_G f \iff \left(\gamma^{-1}(x)I + T\right)\gamma(x)u = T_G f.$$

There exists a unique $v \in L^{2,\frac{1}{2}}(G, \mathcal{C}\ell_{n,0})$ such that $(\gamma^{-1}(x)I + T_G)v = T_G f$ and there exists a unique $u \in L^{2,-\frac{1}{2}}(G, \mathcal{C}\ell_{n,0})$ with $u = \gamma^{-1}(x)v$. Thus

$$u + T_G\gamma(x)u = T_G f$$

has a unique solution u. Moreover, $u = -T_G(\gamma(x)u - f)$ and hence $tr u \in \operatorname{im} Q_\Gamma$ and

$$Du = -\gamma(x)u + f \in L^{2,\frac{1}{2}}(G, \mathcal{C}\ell_{n,0})$$

is well-defined. •

Theorem 7 *Let G be a bounded domain and $\phi(x,u)$ be a monotone, coercive Carathèodory function on $G \times \mathcal{C}\ell_{n,0}$ satisfiyng $(A\alpha)$ with $\alpha = 0$ and K be a linear bounded, positive operator from $L^2(G, \mathcal{C}\ell_{n,0}) \to L^2(G, \mathcal{C}\ell_{n,0})$. Then*

$$\phi u + (\lambda H_G + K)u + Du + \gamma(x)u = g$$

has a unique solution $u \in L^2(G, \mathcal{C}\ell_{n,0})$ for any $g \in L^2(G, \mathcal{C}\ell_{n,0})$ and each $\lambda \in \mathbb{R}^n$.

Proof: Because G is a bounded domain we have $L^{2,-\alpha}(G, \mathcal{C}\ell_{n,0}) \equiv L^{2,\alpha}(G, \mathcal{C}\ell_{n,0}) \equiv L^2(G, \mathcal{C}\ell_{n,0})$. Now, use Theorem 3 and 6.•

Theorem 8 *Let G be a bounded or unbounded smooth domain and $\Phi(x,u)$ be a monotone, coercive Carathèodory function on $G \times \mathcal{C}\ell_{n,0}$ satisfiyng $(A\alpha)$ with $\alpha = -\frac{1}{2}$ and K a linear bounded, positive operator from $L^{2,-\frac{1}{2}}(G, \mathcal{C}\ell_{n,0}) \to L^{2,\frac{1}{2}}(G, \mathcal{C}\ell_{n,0})$. Then*

$$\Phi u + Ku + Du + \gamma(x)u = f$$

has a unique solution $u \in D(A)$ for any $f \in L^{2,\frac{1}{2}}(G, \mathcal{C}\ell_{n,0})$.

Proof: Theorem 3. •

Theorem 9 *The operator $-HD : H^{1,-\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0}) \subset L^{2,-\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0}) \to L^{2,\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$ is a maximal monotone mapping.*

Proof: From Theorem 2 we get the monotonicity of $-HD$ and from Theorem (iv) we know that $-H$ is invertible and the inverse operator is given by H. Using Corollary 2 we get

$$-HDu = f \iff Du = Hf \iff u = TDu = THf.$$

Thus for arbitrary $f \in L^{2,\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$ there exists a uniquely determined $u \in H^{1,-\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$ such that $-HDu = f$. •

Theorem 10 *Let $\Phi(x,u)$ be a monotone, coercive Carathèodory function on $\mathbb{R}^n \times \mathcal{C}\ell_{n,0}$ satisfiyng $(A\alpha)$ with $\alpha = -\frac{1}{2}$ and K a linear bounded, positive operator from $L^{2,-\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0}) \to L^{2,\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$. Then*

$$\Phi u + Ku - HDu = f$$

has a solution $u \in H^{1,-\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$ for any $f \in L^{2,\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$. This solution is unique if Φ or K are strictly monotone.

Proof: Theorem 3 . •

Theorem 11 *For $\mu \in \mathbb{R}$ the operator $\mu D - HD : H^{1,-\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0}) \subset L^{2,-\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0}) \to L^{2,\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$ is a maximal monotone mapping.*

Proof: We have already seen that $\mu D - HD$ is monotone. Now we want to show that there exist an inverse operator. We have

$$\mu Du - HDu = (\mu I - H)Du = f \iff (\mu I - H)v = f \text{ in } L^{2,-\frac{1}{2}}(G, \mathcal{C}\ell_{n,0}).$$

The operator $\mu I - H$ is invertible and the inverse operator is given by

$$\frac{1}{1+\mu^2}(\mu I + H)$$

thus

$$Du = v = \frac{1}{1+\mu^2}(\mu I + H)f \iff u = TDu = \frac{1}{1+\mu^2}(\mu T + TH)f \in H^{1,-\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$$

for any $f \in L^{2,-\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$. •

Theorem 12 *Let $\Phi(x,u)$ be a monotone, coercive Carathèodory function on $\mathbb{R}^n \times \mathcal{C}\ell_{n,0}$ satisfiyng $(A\alpha)$ with $\alpha = -\frac{1}{2}$ and K a linear bounded, positive operator from $L^{2,-\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0}) \to L^{2,\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$. Then*

$$\Phi u + Ku + \mu Du - HDu = f$$

has a solution $u \in H^{1,-\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$ for any $f \in L^{2,\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$ and each $\mu \in \mathbb{R}$. This solution is unique if Φ or K are strictly monotone.

Proof: Theorems 3 and 11. •

Remark: We dealt with the Clifford algebra $\mathcal{C}\ell_{n,0}$, i.e. $e_j^2 = +1$. This seems to be unusual. But the operators H_G and H are not monotone if we use $\mathcal{C}\ell_{0,n}$. Nevertheless the operators iH_G and iH are monotone in spaces over the complexified Clifford algebras $\mathcal{C}\ell_{0,n}(\mathbb{C})$.

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Swanhild Bernstein
Institute of Applied Mathematics I
Department of Mathematics and Computer Science
University of Mining and Technology
D-09596 Freiberg
Germany

e-mail bernstein@mathe.tu-freiberg.de

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