Normal Blow-ups and their Expected Defining Equations

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Introduction

Let R be a noetherian ring and let I be an ideal of R with grade $I > 0$. Given a presentation

$$R^m \xrightarrow{\phi} R^n \longrightarrow I \longrightarrow 0$$

of I, let $\underline{x} = x_1, ..., x_s$ be generators for the ideal $I_1(\phi)$ generated by the entries of ϕ, and let $\underline{T} = T_1, ..., T_n$ be variables over R. We write

$$\underline{T} \cdot \phi = \underline{x} \cdot B(\phi)$$

for some matrix $B = B(\phi)$; we call B a *Jacobian dual* of ϕ. This matrix $B(\phi)$ plays an important role in the study of the polynomial relations amongst the generators of I, or in other words, the defining equations of the *blow-up ring* $\mathcal{R}(I) = R[It] \cong \oplus_{i \geq 0} I^i$. Indeed, we may present the blow-up ring as

$$\mathcal{R} = R[It] \;\cong\; R[T_1, ..., T_n]/Q,$$

such that

$$(\underline{x}B(\phi), I_s(B(\phi))) \subset Q.$$

We say that the defining ideal of $\mathcal{R}$ has *the expected form* if equality holds, i.e. if $Q = (\underline{x}B(\phi), I_s(B(\phi)))$. Of course this holds whenever the defining ideal is generated solely by the linear forms $\underline{x}B$, or equivalently that $\mathcal{R} \cong S(I)$, the symmetric algebra of I. The normality of the blow-up ring for such ideals *of linear type* has been studied by various authors ([HUV],[V1])

In this work, we are interested in determining the normality of $\mathcal{R}$ when its defining ideal has the expected form. It turns out that if $R = k[x_1, ..., x_d]$ is a polynomial ring over a field, and ϕ has linear entries, then fairly generally the blow-up ring is regular in codimension one. Thus such an ideal is normal in the presence of Serre's condition (S_2). As a corollary, we obtain the normality of reduced perfect ideals of codimension 2 with linear presentation, as the defining ideal of the blow-up ring is known to be Cohen-Macaulay and have the expected form [MU].

It can happen, however, that the blow-up rings of such ideals can be considerably smoother. In fact, the blow-up Proj $R[It]$ itself can be smooth. For a perfect ideal of codimension 2, generated by $n = d + 1$ elements, we show that Proj $\mathcal{R}$ is smooth precisely when the corresponding fiber $\mathcal{R} \otimes_R k$ is smooth. Using this, one may construct examples of 4-generated homogeneous prime ideals in $\mathbf{Q}[x, y, z]$ whose blow-up is smooth, thus negatively answering a question of P. Francia.

We also consider the local case, where now (R, m) is a regular local ring of dimension d, I an ideal of R with presentation matrix ϕ such that $I_1(\phi) = m$, such that the defining ideal of $\mathcal{R}$ has the expected form. It turns out that such an ideal need not be normal. However, in the case $n = d + 1$, we show that I is normal whenever the fiber $\mathcal{R} \otimes_R k$ is reduced. This gives a criterion for the normality of 4-generated unmixed ideals of codimension 2 in a 3-dimensional equicharacteristic regular local ring, having a Cohen-Macaulay blow-up ring. This complements recent work of Huckaba and Huneke [HH] which gives on the other hand a structure theorem for normal 4-generated unmixed ideals of codimension 2 in a 3-dimensional equicharacteristic regular local ring having a non-Cohen-Macaulay blow-up ring.

1 Linear Presentation

In this section we study the normality of I in the case of linear presentation. We begin with the following general result.

Proposition 1.1. Let $R = k[x_1, ..., x_d]$ and let I be an ideal with a linear presentation matrix such that the defining ideal of $\mathcal{R}$ has the expected form. If $\mathcal{R}(I_p)$ satisfies (R_1) for every $p \in V(I), p \neq m = (x_1, ..., x_d)$, then $\mathcal{R}$ satisfies (R_1).

Proof. Let $P \in \text{Spec}(\mathcal{R})$ with dim $\mathcal{R}_P \leq 1$ and set $p = P \cap R$. Since $\mathcal{R}_P$ is a localization of $\mathcal{R}(I_p)$, we may assume that $p = m$. Now since I is homogeneous, it is well-known that $\mathcal{R} \otimes_R k$ is a domain, hence $m\mathcal{R}$ is a prime ideal. As $m\mathcal{R} \subset P$, it follows that $P = m\mathcal{R}$, and that dim $\mathcal{R} \otimes_R k =$ dim $\mathcal{R} - ht\ m\mathcal{R} = d$. It suffices to show that $\mathcal{R}_{m\mathcal{R}}$ is regular.

Let ϕ be a matrix with linear entries presenting I and let $B(\phi)$ be its Jacobian dual. Then $B(\phi)$ defines a $k[T]$-module F by

$$k[T]^m \xrightarrow{B(\phi)} k[T]^d \longrightarrow F \longrightarrow 0$$

such that $S_R(I) = S_{k[T]}(F)$.

We now modify an argument from [UV]. From the presentation, we see the ideal $J = I_d(B(\phi))$ annihilates F. Thus tensoring with $Q = k[T]/J \cong \mathcal{R} \otimes_R k$, gives a presentation

$$Q^m \xrightarrow{B(\phi)} Q^d \longrightarrow F \longrightarrow 0$$

of F as a Q-module. On the other hand, if we let $\mathcal{A}$ be the kernel of the natural epimorphism $S(I) \longrightarrow \mathcal{R}(I)$, then by hypothesis the image of J in $S(I)$ generates $\mathcal{A}$. Hence we have isomorphisms

$$\mathcal{R} \cong S(I)/\mathcal{A} \cong S_Q(F).$$

It follows that F is a torsionfree Q-module of linear type, and in particular that dim $S_Q(F)$ = dim $Q + rk_Q F = d + rk_Q F$ ([SV]).

But dim $S_Q(F)$ = dim $\mathcal{R} = d + 1$, hence F has rank one, and is isomorphic to an ideal in Q, necessarily of linear type. Thus if K denotes the field of fractions of R, then $\mathcal{R}_{m\mathcal{R}} \cong S_Q(F)_{(x_1,...,x_d)} \cong S_K(F \otimes_R K)_{(\underline{x})}$ is a localization of a polynomial ring over K, hence regular.

Corollary 1.2. Let $R = k[x_1, ..., x_d]$, let I be an ideal with linear presentation, which is normal on the punctured spectrum, and suppose that $\mathcal{R}$ satisfies (S_2) and that its defining ideal has the expected form.

Then I is normal.

An important case in which one knows the defining ideal of $\mathcal{R}$ has the expected form is for perfect ideals of codimension two. More precisely, if I is a perfect ideal of codimension two with a linear presentation matrix, and if $\mu(I_p) \leq$ dim R_p for all $p \in V(I)$ with $p \neq m = (x_1, ..., x_d)$, then $\mathcal{R}$ is Cohen-Macaulay and its defining ideal has the expected form [MU, 1.3].

Corollary 1.3. Let $R = k[x_1, ..., x_d]$ and let I be an integrally closed perfect ideal of codimension 2, with a linear presentation matrix, satisfying $\mu(I_p) \leq \max\{2, \dim R_p - 1\}$ for every $p \in V(I)$ with dim $R_p < d$.

Then I is normal.

Proof. Since $\mathcal{R}$ is Cohen-Macaulay, and the defining ideal has the expected form, it only remains to show that I is normal on the punctured spectrum. This holds by the condition on the local number of generators (e.g. [UV, 4.2]) and the fact that I is generically normal [G].

We now explore the question of when the blow-up is in fact smooth. We give the following criterion.

Proposition 1.4. Let k be a perfect field, $R = k[x_1, ..., x_d]$, and let I be an ideal generated by $n > d$ elements, with a linear presentation matrix ϕ. Let $B(\phi)$ be the Jacobian dual, let Δ_i denote its maximal minors, and let $\partial\Delta/\partial T$ denote the matrix whose ijth entry is $\partial\Delta_i/\partial T_j$.

(a) If $\sqrt{I_{n-d}(\partial\Delta/\partial T)}$ is the irrelevant ideal of $k[T_1, ..., T_n]$, then Proj $\mathcal{R}$ is smooth.

(b) Suppose in addition that the defining ideal of $\mathcal{R}$ has the expected form, $n = d+1$, and that char $k \not| d$.

Then Proj $\mathcal{R}$ is smooth if and only if $\sqrt{I_1(\partial\Delta/\partial T)}$ is the irrelevant ideal of $k[T_1, ..., T_n]$.

Proof. We compute the Jacobian matrix of the elements $\underline{x}B(\phi), \Delta_i$ of the defining ideal of $\mathcal{R}$:

$$\mathcal{J} = \left(\begin{array}{c|c} B(\phi)^t & \phi^t \\ \hline 0 & \partial\Delta/\partial T \end{array} \right).$$

Since the defining ideal has codimension $n-1$, the singular locus of $\mathcal{R}$ is defined by an ideal containing in particular the $(n-1)$-sized minors of $\mathcal{J}$. But the $(n-1)$-sized minors of $\mathcal{J}$ clearly contains the ideal $I_{d-1}(B(\phi)) \cdot I_{n-d}(\partial\Delta/\partial T)$. Thus $V(I_{n-1}(\mathcal{J})) \subset V(I_{n-d}(\partial\Delta/\partial T)) \cup V(I_{d-1}(B(\phi)))$.

On the other hand, we claim that $I_1(\partial\Delta/\partial T) \subset I_{d-1}(B(\phi))$. This is a consequence of a classical fact about determinants: If $A = (a_{ij})$ is any square matrix whose entries are functions of t, and $\Delta = \det(A)$, then we have

$$\frac{d\Delta}{dt} = \sum_{i,j} \frac{da_{ij}}{dt}\Delta_{ij},$$

where Δ_{ij} denotes the (signed) minor obtained by deleting the ith row and jth column. Thus in particular, the partial derivatives of the maximal minors must lie in the ideal generated by the submaximal minors. Hence in fact we have that $V(I_{n-1}(\mathcal{J})) \subset V(I_{n-d}(\partial\Delta/\partial T))$.

Now in case (a), it follows that if $p \in$ Proj $\mathcal{R}$, then the local ring $\mathcal{R}_p$ is regular. Thus all the local rings of the blow-up are regular, hence Proj $\mathcal{R}$ is smooth.

For the converse (b), suppose now that the defining ideal has the expected form, and that that Proj $\mathcal{R}$ is smooth. This implies that $\sqrt{I_{n-1}(\mathcal{J})}$ contains the irrelevant ideal of $R[T_1, ..., T_n]$, hence for $1 \leq i \leq n$ monomials T_i^k belong to the ideal $I_{n-1}(\mathcal{J})$ for some k. Moreover, we see at least, modulo $(x_1, ..., x_d)$, that the Jacobian ideal is contained in the ideal generated by the maximal minors Δ_i of $B(\phi)$, and their partial derivatives $I_1(\partial\Delta/\partial T)$. But by Euler's formula, $d\Delta_i \in (\partial\Delta_i/\partial T_1, ..., \partial\Delta_i/\partial T_n)$. Thus as d is invertible in k, all the Δ_i belong to $\partial\Delta/\partial T$. Thus modulo $(x_1, ..., x_d)$, the Jacobian ideal is contained in $I_1(\partial\Delta/\partial T)$, and hence $T_i^k \in (x_1, ..., x_d, I_1(\partial\Delta/\partial T))$. But by the homogeneity, $T_i^k \in I_1(\partial\Delta/\partial T)$. Hence $\sqrt{I_1(\partial\Delta/\partial T)}$ is the irrelevant ideal.

Corollary 1.5. Let k be a perfect field with char $k \not| d$, let $R = k[x_1, ..., x_d]$, let ϕ be a $d+1 \times d$ matrix with linear entries satisfying $ht\ I_t(\phi) \geq d-t+2$ for every $2 \leq t \leq d$ and let $I = I_d(\phi)$. Set $F = \det B(\phi)$.

Then Proj $R[It]$ is smooth if and only if $\sqrt{(\partial F/\partial T_1, ..., \partial F/\partial T_{d+1})}$ is the irrelevant ideal.

P. Francia has asked ([OV]) whether a 1-dimensional prime ideal in a regular local with a smooth blow-up is necessarily a complete intersection. We are now able to give a negative answer to this question. To obtain an example in $k[x, y, z]$ with a smooth blow-up we may choose the entries of ϕ to be sufficiently general. But of course, if k is algebraically closed, this ideal is not prime. We contruct a prime ideal over the rationals, using the method employed in [HH, Theorem 2.13].

Example 1.6. Let $R = \mathbf{Q}[x, y, z]$, and let I be the ideal generated by the maximal minors of the matrix

$$\begin{pmatrix} 0 & x-y & y-z \\ z-y & 0 & x-y+z \\ z & 0 & x-2y \\ x & y & z \end{pmatrix}.$$

Then I is a prime ideal of codimension two and Proj $R[It]$ is smooth.

Proof. One may check that the partials of the determinant of the Jacobian dual of this matrix is in fact an irrelevant ideal. Thus Corollary 1.5 implies that the blow-up is smooth.

To prime that I is prime, one verifies that $\mathbf{Q}[z] \subset \mathbf{Q}[x, y, z]/I$ is a Noether normalization, has degree 6 over $\mathbf{Q}[z]$, and that the polynomial $f(x, z) = x^6+4x^5z-15x^4z^2+3z^3+x^2z^4-xz^5$ belongs to $I \cap \mathbf{Q}[x, z]$. Thus [V2, 10.4.19] implies that I is prime if and only if f is irreducible. But f is irreducible if and only if its dehomogenization $1+4z-15z^2+3z^3+z^4-z^5$ is. This latter polynomial is obviously irreducible, e.g. via reduction modulo 2.

A conjecture of the second author [M, 1.2] asserts that, for any perfect Gorenstein ideal of codimension three, of linear type on the punctured spectrum, in an odd dimensional local (Gorenstein) ring, the defining ideal of $\mathcal{R}$ has the expected form. The following gives an example of such an ideal whose blow-up is smooth.

Example 1.7. Let k be a field, let $R = k[x, y, z]$, and let $I = (x^2-y^2, y^2-z^2, xy, yz, xz)$. Then I is a Gorenstein ideal with linear presentation, Proj $R[It]$ is smooth, but $R[It]$ does not satisfy (S_2).

Proof. Note that I is generated by the 4×4 Pfaffians of the alternating matrix

$$\phi = \begin{pmatrix} 0 & 0 & z & 0 & y \\ 0 & 0 & 0 & x & y \\ -z & 0 & 0 & y & x \\ 0 & -x & -y & 0 & -z \\ -y & -y & -x & z & 0 \end{pmatrix}.$$

Hence by the Buchsbaum-Eisenbud structure theorem, I is Gorenstein and ϕ is a presentation matrix of I. The fact that $R[It]$ does not satisfy (S_2) is a consequence of [NV, Theorem 2.6].

To prove Proj $R[It]$ is smooth, we may use Proposition 1.4. Writing $(T_1, ..., T_5)\phi = (xyz)B(\phi)$, we see that the Jacobian dual is the 3×5 matrix

$$B(\phi) = \begin{pmatrix} 0 & -T_4 & -T_5 & T_2 & T_3 \\ -T_5 & -T_5 & -T_4 & T_3 & T_1 + T_2 \\ -T_3 & 0 & T_1 & T_5 & -T_4 \end{pmatrix}.$$

One easily sees that 3 of the 10 maximal minors of $B(\phi)$ are redundant, and the ideal of maximal minors is generated by the 7 cubics

$$T_1T_3T_4 + T_2T_3T_4 - T_3^2T_5 + T_4^2T_5,$$

$$T_3^2T_4 - T_2T_3T_5 - T_4T_5^2,$$

$$T_1T_2T_3 + T_2^2T_3 - T_3^3 + T_2T_4T_5 + T_3T_5^2,$$

$$T_1T_3T_4 - T_1T_2T_5 + T_4^2T_5 - T_5^3,$$

$$T_1^2T_4 + T_1T_2T_4 - T_4^3 - T_1T_3T_5 + T_4T_5^2,$$

$$T_3T_4^2 + T_1T_4T_5 - T_3T_5^2,$$

$$T_1^2T_2 + T_1T_2^2 - T_1T_3^2 - T_2T_4^2 + T_1T_5^2 + T_2T_5^2.$$

A routine verification shows that the 2-sized minors of the matrix of all the partial derivatives generate an irrelevant ideal. Hence the blow-up is smooth.

In fact it holds for this example that the defining ideal of $\mathcal{R}$ has the expected form. This may be verified using the computer algebra system *Macaulay* [BS].

The Grauert-Riemenschneider vanishing theorem (as reformulated by Sancho de Salas [S]) states for an ideal I in a local Cohen-Macaulay ring, essentially of finite type over the complex numbers, with Proj $R[It]$ smooth, the associated graded ring $gr_{I^n}R$ is Cohen-Macaulay for all $n >> 0$. For the previous example, the Grauert-Riemenscheider theorem now implies (at least for $k = \mathbf{C}$) that the associated graded ring of all sufficiently large powers of I is Cohen-Macaulay. Since R is regular, this is equivalent to the Cohen-Macaulayness of the corresponding blow-up ring [L]. In our case, this holds already for the second power. Indeed, $I^n = m^{2n}$ for all $n \geq 2$, where $m = (x, y, z)$, so $R[I^nt]$ is Cohen-Macaulay for all $n \geq 2$ [Va].

The fact that I is not integrally closed would also follow from [CHV, 3.2], as an integrally closed perfect Gorenstein ideal of codimension three in a local Gorenstein ring is a complete intersection.

2 The Local Case

In this section we consider the corresponding problem in the local case. Namely, let (R, m) be a d-dimensional regular local ring, let $x_1, ..., x_d$ be a regular system of parameters, let I be an ideal, and let ϕ be a presentation matrix of I. We assume that $I_1(\phi) = (x_1, ..., x_d)$, so that there exists a (not necessarily unique) Jacobian dual $B(\phi)$, satisfying

$$(T_1, ..., T_n)\phi = (x_1, .., x_d)B(\phi).$$

In addition, we consider now only the case that I is generated by $n = d+1$ elements. We would like to know when $\mathcal{R}$ is normal (or at least satisfies (R_1)) assuming its defining ideal has the expected form.

But unlike the graded case, this need not occur without further assumptions. To see this, let ϕ be a $d+1 \times d$ matrix, with entries in m, whose last row is $x_1 \cdots x_d$, and set $I = I_d(\phi)$. Write

$$\phi = \left(\frac{\psi}{x_1 \cdots x_d} \right)$$

for some matrix ψ. If $ht\ I_t(\phi) \geq d - t + 2$ for every $2 \leq t \leq d$, then it is known that $\mathcal{R}$ is Cohen-Macaulay and its defining ideal has the expected form ([SUV],[MU]). But if all the entries of ψ lie in m^2, then I is not a normal ideal. Indeed, it is easy to see that $m\mathcal{R}$ has a unique minimal prime which is a singular point of $\mathcal{R}$.

From this example, one might expect that I might be normal if ψ contains "sufficiently many" regular parameters. But this appears awkward to make precise. On the other hand, this example is lacking another attribute of the graded case, namely that the fiber $\mathcal{R} \otimes_R k$ is a domain. In fact, $\mathcal{R} \otimes_R k \cong k[T_1, ..., T_{d+1}]/(T_{d+1}^d)$, so the fiber is not even reduced. It turns out that this is necessarily the case.

Theorem 2.1. Let k be a perfect field, let (R, m, k) be a complete equicharacteristic regular local ring of dimension d, with char $k \not| \, d$, and let I be an ideal generated by $n = d+1$ elements, with $I_1(\phi) = m$, such that $\mathcal{R}(I_p)$ satisfies (R_1) for every $p \in V(I), p \neq m$. Suppose further that the defining ideal of $\mathcal{R}$ has the expected form, and that $\mathcal{R} \otimes_R k$ is reduced.

Then $\mathcal{R}$ satisfies (R_1).

Proof. As in the proof of Proposition 1.1, it suffices to show that $\mathcal{R}$ is regular locally at every minimal prime of $m\mathcal{R}$. Now $R = k[[x_1, ..., x_d]]$ and $m = (x_1, ..., x_d)$. Let $B(\phi)$ be the Jacobian dual of ϕ such that the defining ideal of $\mathcal{R}$ is generated by the entries of $(x_1, .., x_d)B(\phi)$ and the maximal minors Δ_i of $B(\phi)$. By assumption, we may write $\mathcal{R} \otimes_R k \cong k[T_1, ..., T_n]/(\delta_1, ..., \delta_r)$, where δ_i are images of Δ_i, and are squarefree polynomials.

We may use the Jacobian criterion to prove regularity. Let $S = R[T_1, ..., T_n]$ and let $\mathcal{J}$ be the Jacobian matrix of the defining ideal of $\mathcal{R}$ with respect to $x_1, ..., x_d$ and $T_1, ..., T_n$. It suffices to show that $I_d(\mathcal{J})$ is not contained in any minimal prime of $m\mathcal{R}$, or equivalently, that

$$ht\ I_d(\overline{\mathcal{J}}) \geq 2,$$

where " $-$ " deontes images in $S \otimes_R k = S/mS$.

But over $S \otimes_R k = k[T_1, ..., T_n]$, the Jacobian has the form

$$\overline{\mathcal{J}} = \left(\begin{array}{c|c} \overline{B}^t & 0 \\ \hline * & \partial\delta/\partial T \end{array} \right).$$

In particular, the d-sized minors contains the product $I_{d-1}(\overline{B})I_1(\partial\delta/\partial T)$. Since the δ_i are maximal minors of $\overline{B}$, as we have seen in the proof of Proposition 1.4, it will be enough to show that the ideal of the partials of the δ_i has height at least two.

Let f be any of the δ_i. We claim that the ideal $(\partial f/\partial T_1, ..., \partial f/\partial T_n)$ has height at least 2. For otherwise, all these partials would have an irreducible factor g of strictly lower degree. But then Euler's formula

$$df = \sum_{i=1}^{n} T_i \partial f/\partial T_i,$$

and the fact that d is a unit implies that $g \mid f$ as well. Write $f = gl$, for some form l. Then differentiation shows that $g \mid l\partial g/\partial T_i$ for every i. As g is irreducible, this means that $g \mid l$ or $g \mid \partial g/\partial T_i$ for every i. Of course the latter condition forces all the partials of g to vanish.

Suppose that $g \not| \ l$. Then $\partial g/\partial T_i = 0$ for every i. This implies that k has positive characteristic p, and that g is a polynomial in T_i^p. Now as k is perfect, we may take pth roots, and write $g = h^p$ for some form h. This contradicts the irreduciblity of g.

Hence we have that $g \mid l$. But then $g^2 \mid f$, which contradicts the fact that f is squarefree. It follows that $ht\ (\partial f/\partial T_1, ..., \partial f/\partial T_n) \geq 2$, which completes the proof.

Corollary 2.2. Let (R, m, k) be a complete equicharacteristic regular local ring of dimension d, with char $k \not| \ d$, and k perfect, and let I be an ideal with $\mu(I) = d + 1$, $I_1(\phi) = m$, which is normal on the punctured spectrum, and suppose further $\mathcal{R}$ satisfies (S_2), that its defining has the expected form, and that $\mathcal{R} \otimes_R k$ is reduced.

Then I is normal.

Recently, Huckaba and Huneke [HH] have characterized the 4-generated unmixed ideals of codimension 2, in a 3-dimensional equicharacteristic regular local ring, whose blow-up ring is normal and not Cohen-Macaulay. We now obtain a criterion for the blow-up ring to be both normal and Cohen-Macaulay.

Corollary 2.3. Let (R, m, k) be a 3-dimensional equicharacteristic regular local ring with k a perfect field of char $k \neq 3$, and let I be an unmixed 4-generated, integrally closed, generic complete intersection ideal of codimension 2 and assume that one row of a minimal presentation matrix ϕ of I generates m, and that $\mathcal{R} \otimes_R k$ is reduced.

Then $\mathcal{R}$ is a normal and Cohen-Macaulay ring.

Proof. We may assume that is R is complete. It is known that $\mathcal{R}$ is Cohen-Macaulay, and that its defining ideal has the expected form ([AH],[SUV],[MU]). Thus the result follows from Corollary 2.2.

We should point that although we assume that $I_1(\phi) = m$, it is almost necessary. For example, by [HH, 2.1], for a normal 4-generated unmixed generic complete intersection of codimension two in a 3-dimensional equicharacteristic regular local ring, $I_1(\phi) = (x, y, z^n)$ for some regular system of parameters x, y, z.

One might also expect that the reducedness of $\mathcal{R} \otimes_R k$ is necessary. This is actually not the case, and we conclude with perhaps the simplest example showing this.

Example 2.4. Let $R = k[[x, y, z]]$, let

$$\phi = \begin{pmatrix} z & y^2 & z \\ 0 & x^2 + z^2 & y^2 \\ 0 & 0 & y + x^2 \\ x & y & z \end{pmatrix},$$

and let $I = I_3(\phi)$.

Then I is a reduced normal perfect ideal of codimension 2, $\mathcal{R}$ is Cohen-Macaulay, but $\mathcal{R} \otimes_R k$ is not reduced.

Proof. We have seen that the blow-up ring is Cohen-Macaulay, and a computation shows that I is reduced. One may take

$$B(\phi) = \begin{pmatrix} T_4 & xT_2 & xT_3 \\ 0 & T_4 + yT_1 & T_3 + yT_2 \\ T_1 & zT_2 & T_4 + T_1 \end{pmatrix}$$

as a Jacobian dual. Then if we set $\Delta = \det B(\phi)$,

$$\Delta = T_4^3 + T_4^2T_1 + yT_4^2T_1 + yT_4T_1^2 - yzT_4T_2^2 - xT_4T_1T_3 - zT_4T_2T_3 + xyT_1T_2^2 - xyT_1^2T_3 + xT_1T_2T_3.$$

It follows $\mathcal{R} \otimes_R k \cong k[T_1, T_2, T_3, T_4]/(T_4^2(T_4 + T_1))$ is not reduced. It remains to show that I is normal. As usual, it suffices to check this locally at each of the two minimal primes of $m\mathcal{R}$.

The Jacobian matrix of the defining ideal of $\mathcal{R}$ has a block modulo m of the form

$$\left(\frac{\overline{B}^t}{\overline{\Delta_x \Delta_y \Delta_z}}\right) = \begin{pmatrix} T_4 & 0 & T_1 \\ 0 & T_4 & 0 \\ 0 & T_3 & T_4 + T_1 \\ T_1T_3(T_4 - T_2) & T_4T_1(T_4 + T_1) & -T_4T_2T_3 \end{pmatrix}.$$

But the maximal minors of this matrix are clearly not contained in either of the two minimal primes of $m\mathcal{R}$. It follows $\mathcal{R}$ satisfies (R_1) and hence that I is normal.

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