ORTHOGONAL PROJECTIONS ON HYPERBOLIC SPACE

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237

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Dedicated to Carlos Berenstein on the occasion of his 60th birthday

Abstract

A well known decomposition of the L^2 space of a bounded domain in the complex plane is extended here to the context of hyperbolic n-space. We will use the model of upper half space with the hyperbolic metric. Applications to boundary value problems for the hyperbolic Laplacian and another Laplace operator are indicated.

Keywords Clifford analysis, hypermonogenic functions, quasi-Cauchy integral formula, Dirac-Hodge equation, hyperbolic harmonic functions

1 Introduction

Function theory for Dirac operators on manifolds have been developed in [C, CC, Cn, M]. In the meantime in a number of papers including [E, EL, EL2, EL1, L] function theory for a hyperbolic Dirac-Hodge operator on upper half n-space endowed with the hyperbolic metric has been developed. In [Q] the authors following [E] develop much of the basic function theory for this operator and the hyperbolic Laplace operator.

In this paper we continue this analysis by showing that a well known decomposition of L^2 integrable functions on a bounded domain in the complex plane in terms of Bergman spaces and Sobolev spaces carry through to the context described here. This may be applied to solving boundary value problems described in [GS] and elsewhere.

2 Preliminaries

Here we will consider upper half space $R^{n,+}$ endowed with the hyperbolic metric $ds^2 = \frac{dx_1^2+\ldots+dx_n^2}{x_n^2}$. With respect to this metric one may consider the adjoint δ to the de Rham exterior derivative d. Namely $\delta = \star d \star$, where $\star$ is the Hodge star map acting on sections in the alternating bundle over R^{n+}. The Dirac- Hodge operator is the differential operator $d+\delta$ acting on differentiable sections on the alternating algebra $\Lambda(R^{n,+})$. The square of $d+\delta$ is the Laplacian $d\delta+\delta d$ with respect to the hyperbolic or Poincare metric. To better understand the Dirac-Hodge operator let us first follow [L] and note that as an vector space the alternating or exterior algebra $\Lambda(R^n)$ is isomorphic to the Clifford algebra Cl_n generated from R^n with negative definite inner product. Namely let us consider $\mathbf{R}^n$ with orthogonal basise $:= \{e_1, \cdots, e_n\}$. Then Cl_n has as its basis

$$1, e_1, \cdots, e_n;\ e_1e_2, \cdots, e_{n-1}e_n;\ \cdots; e_1 \cdots, e_n$$

and $e_1e_j + e_je_i = -2\delta_{ij}$.

We may express the Clifford algebra as $Cl_n = Cl_{n-1} + Cl_{n-1}e_n$ where Cl_{n-1} is the Clifford algebra generated from R^{n-1} with orthonormal basis $e_1, \ldots, e_{n-1}$. So if $A \in Cl_n$ there are unique elements B and $C \in Cl_{n-1}$ such that $A = B + Ce_n$. This gives rise to a pair of maps

$$P : Cl_n \to Cl_{n-1} : P(A) = B$$

and

$$Q : Cl_n \to Cl_{n-1} : Q(A) = C.$$

We will denote $-e_nQ(A)e_n, \in Cl_{n-1}$, by $Q'(A)$.

The Dirac-Hodge operator, $d+\delta$ acting on vector valued sections, now retranslates in Clifford algebra notation as $D + \frac{n-2}{x_n}Q'$, where $D = \Sigma_{j=1}^n e_j \frac{\partial}{\partial x_j}$ is the euclidean Dirac operator. So the Dirac-Hodge equation is defined to be $Df + \frac{n-2}{x_n}Q'(f) = 0$ where $f : U \to Cl_n$ is a differentiable function and U is a domain in $R^{n+} = \{x = x_1e_1 + \ldots = x_ne_n : x_n > 0\}$. See [L] for more details. We shall abbreviate this Dirac equation to $Mf = 0$.

Note, [EL], that if U is a domain in upper half space and $h : U \to Cl_n$ is a C^2 function then

$$-M^2h = \triangle P(h) - \frac{n-2}{x_n}\frac{\partial P(h)}{\partial x_n} + (\triangle Q(h) - \frac{n-2}{x_n}\frac{\partial Q(h)}{\partial x_n} + \frac{n-2}{x_n^2}Q(h))e_n$$

where $\triangle$ is the euclidean Laplacian.

In [A] it is noted for any real valued function $u(x)$ defined on the domain U then $\triangle u - \frac{n-2}{x_n}\frac{\partial u}{\partial x_n}$ is the Laplace, or Laplace-Beltrami, formula for upper half space with respect to the hyperbolic metric. We will denote this Laplacian by $\triangle_{R^{n,+}}$. We will call a Cl_{n-1} valued solution to the equation $\triangle h - \frac{n-2}{x_n}\frac{\partial u}{\partial x_n} = 0$ a hyperbolic harmonic function. It follows that if f is hypermonogenic and C^2 then $P(f)$ is hyperbolic harmonic. Furthermore we shall denote the operator $\triangle - \frac{n-2}{x_n}\frac{\partial}{\partial x_n} + \frac{n-2}{x_n^2}$ by $\triangle^{\star}_{R^{n,+}}$. The equations $\triangle_{R^{n,+}}u = 0$ and $\triangle^{\star}_{R^{n,+}}u = 0$ are both examples of the Weinstein equation. See for instance [Al, GSi, H, L1, W] for details.

Taking $A = a_0 + \ldots a_{1\ldots n}e_1 \ldots e_n \in Cl_n$ we define the norm of A to be, as usual, $\|A\| = (a_0^2 + \ldots + a_{1\ldots n}^2)^{\frac{1}{2}}$. Using the antiautomorphism $- : Cl_n \to Cl_n : -(e_{j_1} \ldots e_{j_r}) = -1^r e_{j_r} \ldots e_{j_1}$ it may be seen that $\|A\|^2$ is the real part of $A\overline{A}$, where $\overline{A}$ is defined to be $-(A)$.

3 Some Integral Formulas

Following [A] it may be seen that $G(x,y) = \int_{\frac{\|x-y\|}{\|x-\hat{y}\|}}^{1} \frac{(1-t^2)^{n-2}}{t^{n-1}}dt$ is a hyperbolic harmonic function. As $G(x,y)$ is real valued then trivially $Q(G(x,y)) = 0$. Consequently $MG(x,y) = DG(x,y)$. Therefore, [L], the function $p(x,y) = DG(x,y)$ is a vector valued hypermonogenic function. Here M and D are acting with respect to the x variable. Following [E, EL2] it may be noted that

$$DG(x,y) = \frac{(1-s^2)^{n-2}}{s^{n-1}}\Big|^{1}_{\frac{\|x-y\|}{\|x-\hat{y}\|}} D\frac{\|x-y\|}{\|x-\hat{y}\|}$$
$$= x_n^{n-2}y_n^{n-1}\left(\frac{(x-y)}{\|x-y\|^n}e_n\frac{(x-\hat{y})}{\|x-\hat{y}\|^n}\right)$$

where $\hat{y} = y_1e_1 + \ldots + y_{n-1}e_{n-1} - y_ne_n$.

Suppose now that U is a domain in upper half space and for two C^1 functions f and g defined on U and taking values in Cl_n we consider the integral $\int_S g(x)\frac{n(x)}{x_n^{n-2}}f(x)d\sigma(x)$, where S is a compact smooth hypersurface lying in U, $n(x)$ is the outer unit normal vector to S at x and σ is the Lebesgue surface measure of S. On assuming that S is the boundary of a bounded subdomain V of U then on applying Stokes' Theorem we obtain

$$\int_S g(x)\frac{n(x)}{x_n^{n-2}}f(x)d\sigma(x) = \int_V ((g(x)D)\frac{1}{x_n^{n-2}}f(x) + g(x)\frac{1}{x_n^{n-2}}Df(x)$$

$$-g(x)\frac{(n-2)}{x_n^{n-1}}e_n f(x))dx^n.$$

It follows that:

Lemma 1 *[EL2] Suppose, f, g, U, S and V are as in the previous paragraph. Then*

$$P\int_S (g(x)\frac{n(x)}{x_n^{n-2}}f(x)d\sigma(x)) = \int_V P[(g(x)M))f(x) + g(x)(Mf(x))]\frac{dx^n}{x_n^{n-2}}.$$

Consequently if $Mf = 0$ $gM = 0$ where $gM = \Sigma_{j=1}^n \frac{\partial g(x)}{\partial x_j}e_j + \frac{n-2}{x_n}Q'(g)$, we have the version of Cauchy's Theorem established in [EL2]. Namely

$$\int_S g(x)\frac{n(x)}{x_n^{n-2}}f(x)d\sigma(x) = 0.$$

It may now be determined that for each $y \in V$

$$P(f(y)) = \frac{1}{\omega_n}P(\int_S p(x,y)\frac{n(x)}{x_n^{n-2}}f(x)d\sigma(x)).$$

This is the Cauchy integral formula arising in [EL2]. Now let us consider $D_yG(x,y)$ where $D_y = \Sigma_{j=1}^n e_j\frac{\partial}{\partial y_j}$. As $\|x-\hat{y}\| = \|y-\hat{x}\|$ then $G(x,y)$ is hyperbolic harmonic in both the variables x and y, and

$$D_yG(x,y) = D_y\int_{\frac{\|y-x\|}{\|y-\hat{x}\|}}\frac{(1-s^2)^{n-2}}{s^{n-1}}ds = h(x,y) = p(y,x)$$

is hypermonogenic in the variable y.

Let M_y denote the hyperbolic Dirac operator with respect to the variable y.

Theorem 1 *[Q] Suppose that ψ is a Cl_{n-1} valued, C^∞ function with compact support on upper half space. Then*

$$M_y\frac{1}{\omega_n}\int_{R^{n,+}} h(x,y)\psi(x)\frac{dx^n}{x_n^{n-2}} = \psi(y).$$

In [E] the kernel $q(x,y) = DH(x,y)$ is introduced where

$$H(x,y) = \frac{1}{(n-2)}\frac{1}{\|x-y\|^{n-2}\|x-\hat{y}\|^{n-2}}.$$

In [E] it is shown that the kernel $q(x,y)$ is the Cauchy kernel for the Q part of a Cauchy Integral Formula for hypermonogenic functions.

From [E] we have

$$f(y) = P(f(y)) + Q(f(y))e_n =$$
$$\frac{2^{n-1}y_n^{n-2}}{\omega_n}(P(\int_{\partial U} r(x,y)\frac{n(x)}{x_n^{n-2}}f(x))d\sigma(x)) - Q(\int_{\partial U} q(x,y)n(x)f(x))d\sigma(x))e_n)$$

where $r(x,y) = y_n^{-n+2}p(x,y)$.

Let us denote the kernel $D_yH(x,y)$ as $s(x,y)$. Following [E] we have that $D_ys(x,y) - \frac{n-2}{y_n}Q'(s(x,y)) = 0$ and $y_n^{n-2}s(x,y)e_n$ is hypermonogenic in the variable y. One may determine the following.

Proposition 1 *Suppose $\psi : U \to Cl_{n-1}e_n$ is a C^1 function. Then for each $y \in U$*

$$\psi(y) = M_y\frac{1}{\omega_n}y_n^{n-2}\int_U s(x,y)\psi(x)dx^n e_n.$$

Now for any $A \in Cl_n$, $P(A) = \frac{1}{2}(A + \hat{A})$ where $\hat{A} = B - Ce_n$ with B and $C \in Cl_{n-1}$. Moreover, $Q(A) = \frac{-e_n}{2}(A - \hat{A})$ and for any elements X and $Y \in Cl_n$ it is straightforward to determine that $\hat{XY} = \hat{X}\hat{Y}$. Using these observations it is noted in [E] that the Cauchy Integral Formula becomes

$$f(y) = \frac{1}{\omega_n}2^{n-1}y_n^{n-2}(\int_{\partial U}\frac{1}{2}(r(x,y)n(x)\frac{n(x)}{x_n^{n-2}}f(x) + \hat{r}(x,y)\frac{\hat{n}(x)}{x_n^{n-2}}\hat{f}(x))d\sigma(x)$$
$$-\frac{e_n}{2}(q(x,y)n(x)f(x) - \hat{q}(x,y)\hat{n}(x)\hat{f}(x))d\sigma(x)).$$

In [E] it is shown that this expression simplifies to

$$f(y) = \frac{2^{n-1}y_n^{n-2}}{\omega_n}(\int_{\partial U}\frac{(x-y)^{-1}n(x)f(x)}{||x-y||^{n-2}||x-\hat{y}||^{n-2}}d\sigma(x)$$
$$-\int_{\partial U}\frac{(\hat{x}-y)^{-1}\hat{n(x)}\hat{f}(x)}{||x-y||^{n-2}||\hat{x}-y||^{n-2}}d\sigma(x)).$$

If we write $E(x,y)$ for $\frac{(x-y)^{-1}}{||x-y||^{n-2}||x-\hat{y}||^{n-2}}$ and $F(x,y)$ for $\frac{(\hat{x}-y)^{-1}}{||x-y||^{n-2}||\hat{x}-y||^{n-2}}$ then this integral formula simplifies to

$$f(y) = \frac{2^{n-1}y_n^{n-2}}{\omega_n}\int_{\partial U}(E(x,y)n(x)f(x) - F(x,y)\hat{n}(x)\hat{f}(x))d\sigma(x).$$

In [Q] it is shown that:

Theorem 2 *Suppose that* $\psi \in C^1(\overline{U})$ *and* $y \in U$ *then*

$$\psi(y) = M_y y_n^{n-2}\left(\int_U E(x,y)\psi(x)dx^n - \int_U F(x,y)\hat{\psi}(x)dx^n\right).$$

It is also shown in [Q] that:

Theorem 3 *Suppose that* S *is a compact, strongly Lipschitz surface lying in upper half space. Suppose also that* S *is the boundary of a bounded domain* U^+ *and an exterior domain* $U^- \subset R^{N,+}$. *Then for each function* $\phi \in L^p(S)$ *for* $1 < p < \infty$ *and for a path* $y_\pm(t) \in U^\pm$ *with nontangential limit* $y(1) = y \in S$ *we have*

$$\lim_{t\to 1} \frac{2^{n-2}y_\pm(t)^{n-2}}{\omega_n} \int_S (E(x,y(t))n(x)\phi(x) - F(x,y(t))\hat{n}(x)\hat{\phi}(x))d\sigma(x)$$

$$= \pm\frac{1}{2}\phi(y) + \frac{2^{n-2}}{\omega_n} PV \int_S y_n^{n-2}(E(x,y)n(x)\phi(x) - F(x,y)\hat{n}(x)\hat{\phi}(x))d\sigma(x)$$

for almost all $y \in S$.

We shall denote the singular integral

$$\frac{2^{n-2}y_n^{n-2}}{\omega_n} PV \int_S (E(x,y)n(x)\psi(x) - F(x,y)\hat{n}(x)\hat{\psi}(x))d\sigma(x)$$

by $T_S(\psi)$.

A minor adaptation of the proof of Theorem 17 in [E] tells us the following:

Theorem 4 *[Q] Suppose* S *is a Lipschitz surface lying in the closure of upper half space and* $\phi \in L^p(S)$ *for some* $p \in (1,\infty)$ *then the integral*

$$\frac{2^{n-2}y_n^{n-2}}{\omega_n} \int_S (E(x,y)n(x)\phi(x) - F(x,y)\hat{n}(x)\hat{\phi}(x))d\sigma(x)$$

defines a left hypermonogenic function $f(y)$ *on* $R^{n,+}\backslash S$.

As $\lim_{y_n\to\infty} y_n^{n-2}E(x,y) = 0$ and $\lim_{y_n\to\infty} y_n^{n-2}F(x,y) = 0$ for each $x \in S$ and $\lim_{y_n\to 0} y_n^{n-2}E(x,y) = \lim_{y_n\to 0} y_n^{n-2}F(x,y) = 0$ for each $x \in S$ then $lim_{y_n\to\infty} f(y) = \lim_{y_n\to 0} f(y) = 0$. It now follows that the operators $\frac{1}{2}I \pm T_S : L^p(S) \to L^p(S)$ are projection operators with images the Hardy spaces

$H^p(U^\pm) = \{f : U^\pm \to Cl_n : f$ is left hypermonogenic and non-tangentially approaches some element in $L^p(S)\}$.

Consequently
$$L^p(S) = H^p(U^+) \oplus H^p(U^-).$$

The operators $\frac{1}{2} \pm T_S$ are generalizations of the Plemelj projection operators to the context of hypermonogenic functions.

4 Orthogonal Projections

Let us consider a bounded domain U with strongly Lipschitz boundary ∂U. Further U and its boundary lie in upper half space. Let $L^2(U, Cl_{n-1})$ denote the space of L^2 integrable, Cl_{n-1} valued functions defined on U. We will first work with the inner product $P(\int_U f(x)\overline{g}(x)\frac{dx^n}{x_n^{n-2}})$. We want a description of the subspace of $L^2(U, Cl_{n-1})$ consiting of all those functions that are orthogonal to square integrable, right hypermonogenic functions with respect to this inner product. Clearly for each y belonging to the complement of $U \cup \partial U$ the function $p(x,y)$ is square integrable over U and is right hypermonogenic in x. So if ψ is orthogonal with respect to $<,>$ to all square integrable hypermonogenic functions defined on U then $\frac{1}{\omega_n}\int_U p(x,y)\overline{\psi}(x)\frac{dx^n}{x_n^{n-2}} = 0$.

But from Theorem 1 we have
$$\overline{\psi}(w) = M_w\frac{1}{\omega_n}\int_U p(w,x)\overline{\psi}(x)\frac{dx^n}{x_n^{n-2}}.$$

So
$$0 = (\frac{1}{\omega_n})^2\int_U p(w,y)M_w\int_U p(w,x)\overline{\psi}(x)\frac{dx^n}{x_n^{n-2}}\frac{dw^n}{w_n^{n-2}}.$$
By Lemma 1 this gives us
$$(\frac{1}{\omega_n})^2 P(\int_{\partial U} p(w,y)\frac{n(w)}{w_n^{n-2}}\int_U p(w,x)\overline{\psi}(x)\frac{dx^n}{x_n^{n-2}}d\sigma(w)) = 0.$$
We will denote the function $\frac{1}{\omega_n}\int_U p(w,x)\overline{\psi}(x)\frac{dx^n}{x_n^{n-2}}$ by $g(w)$. So we have
$$\frac{1}{\omega_n}P(\int_{\partial U} p(w,y)\frac{n(w)}{w_n^{n-2}}g(w)d\sigma(w)) = 0$$
for each y in the complement of $U \cup \partial U$. Consequently if we allow y to move along a path which approaches ∂U non-tangentially we have that $P(-\frac{1}{2}g(y) + \frac{1}{\omega_n}P.V.\int_{\partial U} p(w,y)\frac{n(w)}{w_n^{n-2}}g(w)d\sigma(w)) = 0$ for almost all $y \in \partial U$.

Now let us introduce the function

$$f_g(y) = P(\frac{1}{\omega_n}\int_{\partial U} p(w,y)\frac{n(w)}{w_n^{n-2}}g(w)d\sigma(w))$$

defined on U. Clearly as y moves along a path that approaches the boundary of U non-tangentially we get as boundary value of $f_g(y)$ the expression

$$\frac{1}{2}f_g(y) + P(\frac{1}{\omega_n}P.V.\int_{\partial U} p(w,y)\frac{n(w)}{w_n^{n-2}}f_g(w)d\sigma(w))$$

almost everywhere. Consequently the function $g(x) - f_g(x)$ has the property that $P(g(x) - f_g(x)) = 0$ on ∂U.

Following Proposition 1 we can now use similar arguments to those we have just used to obtain the following:

Lemma 2 *Suppose that $\theta : U \to e_nCl_{n-1}$ is a C^1 function then for each $y \in U$*

$$\overline{\theta}(y) = -Q(M_y\frac{y_n^{n-2}}{\omega_n}\int_U s(x,y)\overline{\theta}(x)dx^n)e_n.$$

So

$$0 = Q((\frac{1}{\omega_n})^2\int_U s(x,y)M_y y_n^{n-2}\int_U s(w,y)\overline{\theta}(w)dw^n dy^n).$$

Consequently

$$0 = Q(\frac{1}{\omega_n}\int_{\partial U} s(x,y)n(y)h(y)d\sigma(y))$$

where

$$h(y) = \frac{y_n^{n-2}}{\omega_n}\int_U s(w,y)\overline{\theta}(w)dw^n.$$

Let $F_h(y) = Q(\frac{y_n^{n-2}}{\omega_n}\int_{\partial U} s(w,y)n(w)h(w)dw^n)$.

Consequently the function $h(x) - F_h(x)$ has the property that $Q(h - F_h) = 0$ almost everywhere on ∂U. So $h(x) - F_h$ has the property that it is orthogonal to any L^2 hypermonogenic function defined on U with respect to the inner product $Q(\int_U f(x)\overline{g}(x)dx^n$.

The previous calculations leads us to define an inner product $< f, g >$, for square integrable functions F and g taking values in Cl_n, to be

$$P(\int_U f(x)\overline{g}(x)\frac{dx^n}{x_n^{n-2}}) + Q(\int_U f(x)\overline{g}(x)dx^n).$$

We shall denote this inner product space by $L^2_{<,>}(U, Cl_n)$. Further the Bergman space $\{f : U \to Cl_n : fM = 0 \text{ and } f \in L^2_{<,>}(U, Cl_n)\}$ is denoted by $B_{<,>}(U)$.

From Theorem 4 we know that $P(f_g)(x) + Q(F_h)(x)e_n$ is a hypermonogenic function on U. We denote this function by $\Theta(\mu)$ where $\mu(x) = \psi(x) + \theta(x)$.

So if $f(x) \in L^2_{<,>}(U, Cl_n)$ is such that $< f, g >= 0$ for each $g \in B_{<,>}(U)$ then for each y in the complement of $U \cup \partial U$ we have

$$\frac{2^{n-1}y_n^{2-n}}{\omega_n}(P(\int_{\partial U} r(x,y)\frac{n(x)}{x_n^{n-2}}\psi(x)d\sigma(x)) - Q(\int_{\partial U} q(x,y)n(x)\psi(x)d\sigma(x) = 0 \tag{1}$$

where $r(x, y) = y_n^{2-n}p(x, y)$ and

$$\psi(y) = y_n^{n-2}(\int_U E(x,y)f(x)dx^{n-2} - \int_U F(x,y)\hat{f}(x)dx^n)$$

.

Following arguments given in [E] it may be seen that equation (1) simplifies to

$$\frac{2^{n-1}y_n^{n-2}}{\omega_n}\int_{\partial U}(E(x,y)n(x)\psi(x) - F(x,y)\hat{n}(x)\hat{\psi}(x))d\sigma(x) = 0.$$

Now if we let y approach the boundary of U non-tangentially we get from the Plemelj formulas that

$$\frac{1}{2}\psi(y) = \frac{2^{n-1}y_n^{n-2}}{\omega_n}P.V.\int_{\partial U}(E(x,y)n(x)\psi(x) - F(x,y)\hat{n}(x)\hat{\psi}(x))d\sigma(x)$$

almost everywhere on ∂U. Consequently

$$\psi(w) - \frac{2^{n-1}w_n^{n-2}}{\omega_n}\int_{\partial U} E(x,w)n(x)\psi(x) - F(x,w)\hat{n}(x)\hat{\psi}(x))d\sigma(x)$$

defines a function Ψ on U satisfying
(i) $\Psi(w) = 0$ on ∂U.
(ii) $M\Psi(w) = f(w)$

So we have established

Theorem 5

$$L^2(U, Cl_n) = B_{<,>}(U) \oplus MH(U)$$

where $H(U) = \{r : U \to Cl_n : r|_{\partial U} = 0 \text{ and } Mr \in L^2(U, Cl_n)\}$.

It follows that the methods described in [GS] for tackling boundary value problems can now be readily adapted to the context described here. In particular one can now handle in a similar way to those methods used in [GS] boundary problems for the operators $\triangle_{R^n,+}$ and $\triangle^*_{R^n,+}$. First in order to solve the equation $\triangle_{R^n,+}u = f$, where f is a bounded, Cl_{n-1} valued, C^1 function defined on a bounded domain U, one solution is given by $u = P(I^2 f)$, where

$$I(f)(y) = y_n^{n-2} \int_U (E(x,y)f(x) - F(x,y)\hat{f}(x))dx^n.$$

See [Q] for details. If we impose the further condition that $u|_{\partial U} = 0$ then we simply apply the constructions obtained in this section to produce the solution $P(I^2(f) - \Theta(I(f))$. This is in complete analogy to arguments developed in [GS] for the euclidean Laplacian.

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