

**Piecewise-Smooth Surfaces as the Union of
Geodesic Disks**

by

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We are motivated by the following definition and question of G. Fejes Tóth:

A **geodesic disk** (figure 1) is a piecewise smooth disk D embedded in \mathbb{R}^3 such that

there exist a center $c \in D$, and radius $r \in \mathbb{R}$, $r > 0$, such that

(a) for all $x \in \partial D$, $d(x, c) = r$ and

(b) for all $x \in \text{int}(D)$, $d(x, c) < r$,

where d is the extrinsic metric on D

(i.e. distance measured along shortest geodesics on D).

Question: *Does there exist a convex body in \mathbb{R}^3 such that ∂B is the union of three geodesic disks with disjoint interiors.*

We will show the answer is no, at least if we assume the convex body is piecewise smooth. We give this as a corollary to a more general theorem. Before stating the theorem we first need another definition:

Let \mathcal{F} be a compact piecewise smooth surface embedded in \mathbb{R}^3 . For any point x on \mathcal{F} we define the **local excess curvature** of \mathcal{F} at x $k(x)$ as follows (figure 2): Since \mathcal{F} is piecewise smooth, for $x \in \mathcal{F}$, x is incident to some finite collection $\{\mathcal{F}_i\}$ of smooth subsets of \mathcal{F} . Each $T_x \mathcal{F}_i$ is a sector of a plane centered at x with angle measure a_i .

Then for $x \in \partial \mathcal{F}$, we define $k(x)$ as:

$$k(x) = \sum a_i - \pi$$

And for $x \in \text{int}(\mathcal{F})$, we define $k(x)$ as:

$$k(x) = \sum a_i - 2\pi$$

Since \mathcal{F} is the union of finitely many smooth pieces, and the boundaries of these smooth pieces are themselves piecewise smooth, the collection of points in \mathcal{F} at which \mathcal{F} is not smooth forms a finite graph, Γ , where the edges are smooth curves in \mathcal{F} .

It is helpful to note that for any $x \in \mathcal{F}$ that is not a vertex of Γ , $k(x) = 0$: This is trivial for x in the interior of a smooth subset of \mathcal{F} . If $x \in \text{int}\mathcal{F}$ and x is in the interior of an edge of Γ , then x is in the interior of a smooth boundary between two smooth subsets of \mathcal{F} , and $k(x) = \pi + \pi - 2\pi$. Finally if $x \in \partial\mathcal{F}$ but x is in the interior of an edge of Γ , then x is on the smooth boundary of a smooth subset of \mathcal{F} and so $k(x) = \pi - \pi$.

We thus define the **excess curvature** $k(X)$ on a subset X of a compact piecewise smooth surface \mathcal{F} embedded in \mathbb{R}^3 to be:

$$k(G) = \sum_{x \in V} k(x)$$

where V is the (finite) set of vertices of $G \cap X$.

We can now state our main theorem:

Theorem 1 *Let \mathcal{F} be a compact piecewise smooth surface embedded in \mathbb{R}^3 , such that \mathcal{F} is the union of n geodesic disks C_i . Then the following inequality holds:*

$$k(G) \geq 2\pi(n - \chi(\mathcal{F}))$$

where G is the graph of points given by $G = \mathcal{F} - \cup C_i$

This immediately gives us two corollaries:

Corollary 2 *Let B be a convex body in \mathbb{R}^3 with piecewise smooth boundary. Then ∂B is the union of at most two geodesic disks.*

This follows immediately from the observation that if B is convex, then $k(x) \leq 0$ for all $x \in \partial B$ and that ∂B must be a topological sphere.

Corollary 3 *Let \mathcal{F} be a smooth surface embedded in \mathbb{R}^3 such that \mathcal{F} is the union of n geodesic disks. Then either \mathcal{F} is a topological sphere and $n = 2$ or \mathcal{F} is itself a geodesic disk and $n = 1$.*

This follows immediately from the observation that if \mathcal{F} is smooth, $k(x) = 0$ for all $x \in \mathcal{F}$. Hence $n = \chi(\mathcal{F})$.

The theorem follows quickly from the following lemma:

Lemma 4 *Let C be a geodesic disk. Then for all $x \in \partial C$, $k(x) \geq 0$*

We will show that the theorem follows from the lemma, and then prove the lemma.

Proof of Theorem 1 Let \mathcal{F} be a compact piecewise smooth surface embedded in \mathbb{R}^3 , such that \mathcal{F} is the union of n geodesic disks. Let G be the finite graph of the boundaries between these disks. We can subdivide the edges of G so that for all $x \in G$, $k(x) = 0$ if x is not a vertex of G . The complement of G has n components. Let V be the set of vertices of G ; let v, e be the number of vertices, edges of G .

For every $x \in V$, let q_x be the number of edges of G meeting x . Then the lemma gives, for all $x \in V$,

$$k(x) > q_x\pi - 2\pi$$

(Note that this is satisfied whether x is in the interior of \mathcal{F} or on the boundary of \mathcal{F}).

Thus:

$$\begin{aligned} \sum_{x \in V} k(x) &\geq \sum_{x \in V} (q_x\pi - 2\pi) \\ &= 2\pi(e - v) \\ &= 2\pi(n - \chi(\mathcal{F})) \end{aligned}$$

□

We now turn to the proof of the lemma.

Proof of the lemma:

Let C be a geodesic disk with center c and radius r .

Since C is piecewise smooth, we can linearly approximate to arbitrary degree a sufficiently small neighborhood of $y \in C$. In particular, in a sufficiently small neighborhood of y , C is arbitrarily close to the union of flat sectors with apex at y , and geodesics are arbitrarily close to straight lines (see figure 3).

Note $d(a, y_1) < d(a, y)$, whence,

$$\begin{aligned}d(c, y_1) &\leq d + d(a, y_1) \\ &< d + d(a, y) \\ &= d(c, y)\end{aligned}$$

This contradicts the assumption that $y, y_1 \in \partial C$.

□

Reference

G. Fejes Tóth, Problem 12b, Discrete Geometry Tagungsbericht 20/1997, Math. Forschungsinstitut Oberwolfach.

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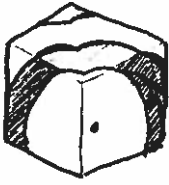


figure 1
concentric geodesic disks

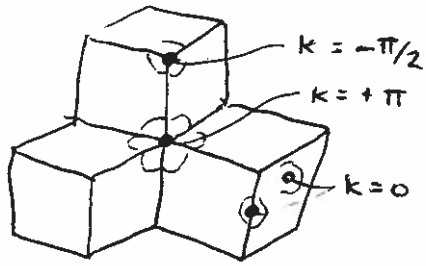


figure 2
local excess curvature

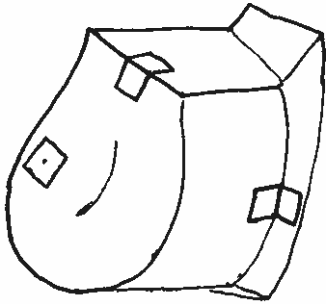


fig 3
approximating linearly

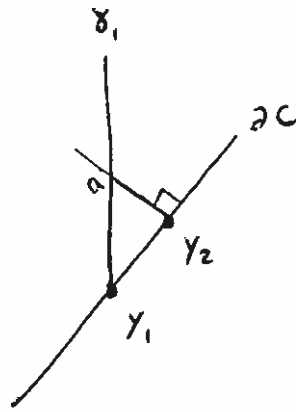


figure 4

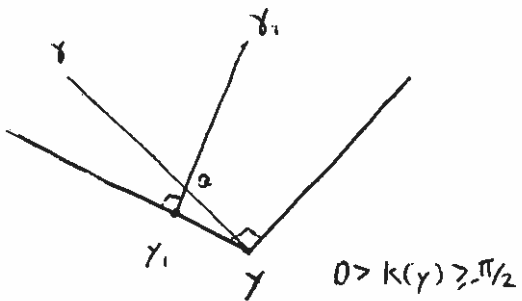


figure 5

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