On sums of sets of primes with positive relative density

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History:

Roth’s theorem. Let $\delta > 0$ and let $A \subset \{1, \ldots, N\}$ such that $|A| \geq \delta N$. Then provided that $N > N(\delta)$, $A$ must contain an arithmetic progression of length three.

Roth’s theorem in the primes. (Green) Let $\delta > 0$ and let $A \subset P_n := \{p \leq n\}$ such that $|A| \geq \delta |P_n|$. Then, provided that $n > n(\delta)$, $A$ must contain an arithmetic progression of length three.
We define

\[ A + A := \{a + a' : a, a' \in A\}. \]

**Theorem 1 (H-Łaba)** Suppose that \( W \) is a random subset of \( \mathbb{Z}/N\mathbb{Z} \) such that each \( x \in \mathbb{Z}/N\mathbb{Z} \) belongs to \( W \) independently with probability \( p \in (CN^{-\theta}, 1] \) where \( 0 < \theta \ll 1 \). Fix \( \delta > 0 \) and suppose \( A \subset W \) such that \( |A| \geq \delta|W| \). Then for every \( \alpha < \delta \),

\[ |A + A| \geq \alpha N \]

with probability \( 1 - o(1) \) as \( N \to \infty \).

**Question:** Can we prove an analogue of this for subsets of the primes?
Example: Suppose that
\[ m = 2 \cdot 3 \cdot 5 \cdots p_t. \]
Define
\[ B := \{ p \in \mathcal{P}_n : p \equiv 1 \pmod{m} \}. \]
The prime number theorem for arithmetic progressions \( \Rightarrow \)
\[ |B| = \frac{n}{\phi(m) \log n} + O\left(\frac{n}{\log^2 n}\right). \]
If \( n \) is sufficiently large, we have
\[ |B| \geq \frac{n}{2\phi(m) \log n}. \]
Define
\[ \delta := \frac{1}{2\phi(m)}. \]
On the other hand,

\[ B + B \subset \{ s \leq 2n : s \equiv 2 \pmod{m} \} \]

and so

\[ |B + B| \ll \frac{n}{m} \sim \frac{\delta n}{\log \log(1/\delta)}. \]

**Theorem 2** (Chipeniuk-H) Let \( \delta > 0 \). Let \( A \subset \mathcal{P}_n \) so that

\[ |A| \geq \delta |\mathcal{P}_n|. \]

Then, assuming \( n \) is sufficiently large (depending on \( \delta \))

\[ |A + A| \geq c_1 \delta e^{-c_2 \sqrt{\log(1/\delta)}} n, \]

where \( c_1 \) and \( c_2 \) are absolute constants.
Definitions

- **Discrete Fourier transform:**
  \[ \hat{f}(\xi) := \frac{1}{N} \sum_{x \in \mathbb{Z}/N\mathbb{Z}} f(x) e^{-2\pi i x \xi / N} \]

- **Convolution:** \((f, g : \mathbb{Z}/N\mathbb{Z} \rightarrow \mathbb{C})\)
  \[ f \ast g(x) := \sum_{y \in \mathbb{Z}/N\mathbb{Z}} f(y) g(x - y) \]

- **\(\|f\|_p\):**
  \[ \|f\|_p := \left( \sum_x |f(x)|^p \right)^{1/p} \]
In the random setting:

\[ W \subset \mathbb{Z}/N\mathbb{Z} : \quad |W| = N^{1-\theta}, \quad 0 < \theta \ll 1 \]

and

\[ |\hat{1}_W(\xi)| \leq N^{-10\theta} \text{ for all } \xi \neq 0. \]

Further, assume

\[ A \subset W \text{ so that } |A| \geq \delta|W|. \]

We define

\[ f(x) := N^{\theta}1_A(x) \]

so that

\[ \frac{1}{N} \sum_{x \in \mathbb{Z}/N\mathbb{Z}} f(x) = \delta. \]
Then, to show that

$$|A + A| \geq \alpha N$$

we show

$$|\{x \in \mathbb{Z}/N\mathbb{Z} : f \ast f(x) > 0\}| \geq \alpha N$$

**Two ingredients**

1. Estimate of Tao-Vu:

   $$\|\hat{f}\|_q \leq M$$ for some $2 < q < 3$

2. Decomposition of Green-Tao: $f = f_1 + f_2$
   where $f_1$ is bounded (anti-uniform) and $f_2$ is 'random' (uniform).
Decomposition: (pick $\epsilon > 0$ later) Let

$$\Lambda := \{\xi \in \mathbb{Z}/N\mathbb{Z} : |\hat{f}(\xi)| \geq \epsilon\},$$

$$B := \{x \in \mathbb{Z}/N\mathbb{Z} : |1 - e^{-2\pi i x \xi/N}| \leq \epsilon \ \forall \ \xi \in \Lambda\},$$

$$f_1(x) := \frac{1}{|B|^2} \sum_{y_1, y_2 \in B} f(x + y_1 - y_2)$$

and

$$f_2 := f - f_1.$$

We can show that this decomposition satisfies:

- $0 \leq f_1 \leq 1 + O(\eta)$

- $\frac{1}{N} \sum_x f_1(x) = \frac{1}{N} \sum_x f(x) \geq \delta$

- $\|\hat{f}_2\|_{\infty} \leq c\epsilon$

- $|\hat{f}_i(\xi)| \leq |\hat{f}(\xi)| \ \forall \ \xi \in \mathbb{Z}/N\mathbb{Z}, \ i = 1, 2.$
We would like to show:

$$|\{x \in \mathbb{Z}/N\mathbb{Z} : f \ast f(x) > 0\}| \geq (\delta - 10\sigma)N.$$  

**Main term:**

$$\left| \{x \in \mathbb{Z}/N\mathbb{Z} : f_1 \ast f_1(x) \geq \sigma \delta N \} \right| > (\delta - 3\sigma)N$$

**Error terms:**

$$\left| \{x \in \mathbb{Z}/N\mathbb{Z} : |f_2 \ast f_i(x)| \geq \frac{\delta \sigma}{10} N \} \right| < \sigma N$$

It suffices to prove:

$$\|f_i \ast f_2\|_2^2 \leq \frac{\sigma^2 \delta^2}{200} \sigma N^3.$$  

Using Hölder’s and Fourier inversion we can show that

$$\|f_i \ast f_2\|_2^2 \leq N^3 \|\hat{f}_2\|_\infty^{1/5} \|\hat{f}_i\|_{19/9}^2 \|\hat{f}_2\|_{19/9}^{9/5}$$
Back to subsets of the primes:

- Let $m = \prod_{p \leq W} p$ where $W$ depends on delta and $W \ll \log \log n$.

- Partition $A$ into residue classes

  $$A^{(b)} := \{a \in A : a \equiv b \pmod{m}\}.$$

  Define

  $$\delta_b := \frac{|A^{(b)}|}{|P_n^{(b)}|}.$$

  We say $b$ is good if $\delta_b \geq \delta/2$. 
• Embed primes in $\mathbb{Z}/N\mathbb{Z}$ where $N \sim n/m$: If $xm + b \in A^{(b)}$, then we map $xm + b \mapsto x$.

Then

$$|A^{(b_1)} + A^{(b_2)}| \geq |A^{(b_1)} + A^{(b_2)}|.$$ 

• Use 'pseudorandomness' to show that

$$|A^{(b_1)} + A^{(b_2)}|$$

large.

• Sum over all pairs $(b_i, b_j)$ to get a lower bound on $A + A$. 

Showing $A_{N}^{(b_1)} + A_{N}^{(b_2)}$ is large:

Green’s modified von Mangoldt function:

$$\lambda_{b,m,N}(x)$$

$$= \begin{cases} 
\frac{\phi(m)}{mN} \log(mx + b) & \text{if } x \leq N \text{ and } mx + b \text{ is prime}, \\
0 & \text{otherwise} 
\end{cases}$$

Define

$$f_b := N 1_{A_{N}^{(b)}} \lambda_{b,m,N}$$

and

$$\nu_b := N \lambda_{b,m,N}.$$ 

Remark: The function $\nu_b$ is 'pseudorandom'.
Discrete majorant property (Green) Let $s > 2$. Then there is a constant $C(s)$ such that

$$\|\hat{f}_b\|_s \leq C(s).$$

Using the decomposition described above, we show that

$$|\{x \in \mathbb{Z}/N\mathbb{Z} : f_{b_1} * f_{b_2}(x) > 0\}| \gg \left(\frac{\delta_{b_1} + \delta_{b_2}}{2} - \epsilon\right)N$$

using methods similar to those described in the random setting.