Some Extremal Algebras For Hermitians

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Abstract. We study three extremal Banach algebras: (a) generated by two hermitian unitaries; (b) generated by an element of norm 1 all of whose odd positive powers are hermitian; (c) generated by an element of norm 1 all of whose even positive powers are hermitian. In all three cases the numerical range is found for various elements. The second algebra is shown to be isometrically isomorphic to a subalgebra of the first. The third algebra is identified with a space of functions.

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1. Introduction. The extremal Banach algebra generated by a single normalised hermitian has been studied extensively (see [4], [5], [6], [7], [8]). Its dual space is a Banach space of entire functions with exponential growth and, consequently, most of the deeper properties come down to delicate questions about such entire functions, often coefficient problems. For two commuting hermitians, the extremal algebra is just the projective tensor product of the two one generator algebras (see [5]). We begin here a study of the case of two non-commuting hermitians. For hermitian elements h, k, the Jordan product $hk + kh$ fails to be hermitian in general, whereas the imaginary Lie product $i(hk - kh)$ is always hermitian. Hence the extremal algebra on two non-commuting hermitians will contain amongst its hermitians the free Lie algebra under this imaginary Lie product. This is a very large complicated algebra, and we shall consider here only the quotient algebra when the generators are idempotents. We get the same algebra by replacing the two hermitian idempotents with two hermitian unitaries; we take this viewpoint since it has nicer algebraic features. Our study is thus reduced here to a very tractable algebra.

Let $J = [-1, 1]$. In Section 2 we study the extremal algebra $Ea(J^2; unit)$ on two hermitian unitaries u, v. There is a natural representation for this algebra as 2×2 matrices of functions (see [10]). We calculate the norm and the numerical range for various elements. The hermitian element $k = \frac{1}{2}i(uv - vu)$ is obviously of special interest here. In fact, k^n is hermitian for all odd $n \in \mathbb{N}$ (but for no even $n \in \mathbb{N}$). We show in Section 3 that the subalgebra of $Ea(J^2; unit)$ generated by k (and 1) is $Ea(J; odd)$, the extremal algebra generated by a normalised hermitian all of whose odd positive powers are hermitian. It is then natural to ask for $Ea(J; even)$, the extremal algebra generated by a normalised element all of whose even positive powers are hermitian; this is a much easier task and is dealt with in Section 4. If all powers of h are hermitian, then the extremal algebra is $C[-1, 1]$ by the Vidav-Palmer theorem.

On the face of it, we might consider hermitian generators of order other than 2. In fact, no cases arise other than of order 2. Suppose first that h is hermitian and $P(h) = 0$ for some polynomial P. Since h has real spectrum, every factor $h - \zeta$ of $P(h)$, with ζ not real, is invertible. Thus, without loss of generality, we may suppose that P has only real roots. Now let $h^n = 1$ with $n \in \mathbb{N}$, n odd. The only real factor of $h^n - 1$ is $h - 1$, and hence $h = 1$. Now let $h^n = 1$ with $n \in \mathbb{N}$, n even. The only real factors of $h^n - 1$ are $h - 1$ and $h + 1$, and hence $h^2 = 1$.

We shall write $\mathbb{D}^-$, $\mathbb{T}$ and $\mathbb{H}^-$ for the sets

$$\{z \in \mathbb{C} : |z| \leq 1\}, \quad \{z \in \mathbb{C} : |z| = 1\}, \quad \{z \in \mathbb{C} : |z - \tfrac{1}{2}| \leq \tfrac{1}{2}\},$$

respectively. For a Banach algebra element a, we denote the spectrum by $\mathrm{Sp}(a)$ and the spectral radius by $r(a)$. We use $|\cdot|_\infty$ to denote the supremum norm, taken over appropriate compact sets. For standard notation and results on numerical ranges, the reader is referred to **[1]** and **[2]**.

2. The extremal algebra $Ea(J^2; unit)$. Let $Ea(J^2; unit)$ denote the extremal Banach algebra on two generators u, v with $u^2 = v^2 = 1$ and u, v hermitian. We note that $\|u\| = \|v\| = 1$, since the spectral radius agrees with the norm for any hermitian. Then $\mathbb{Z}_2 * \mathbb{Z}_2$, the free product of the group $\mathbb{Z}_2$ with itself, is given by

$$\mathcal{I}_2 = \{1, u, v, uv, vu, uvu, vuv, \dots\}.$$

Write $x = uv$. Then, as is well known, $\mathcal{I}_2$ may be regarded as the infinite dihedral group generated by u, x subject to $u^2 = 1$, $ux = x^{-1}u$. Give the group algebra $\mathbb{C}[\mathcal{I}_2]$ its usual involution: $(\sum \alpha_n g_n)^* = \sum \overline{\alpha_n} g_n^{-1}$. Let $\mathcal{J}$ be the subgroup of $\mathbb{C}[\mathcal{I}_2]$ consisting of all finite products of

$$\cos\tau + i(\sin\tau)u, \quad \cos\tau + i(\sin\tau)v \quad (\tau \in \mathbb{R}).$$

Define a norm on $\mathbb{C}[\mathcal{I}_2]$ by

$$\|a\| = \inf\{\sum|\alpha_j| : \sum\alpha_j a_j = a,\ \alpha_j \in \mathbb{C},\ a_j \in \mathcal{J}\}.$$

Then, since $\exp(i\tau u) = \cos\tau + i(\sin\tau)u$ $(\tau \in \mathbb{R})$, we have:

THEOREM 2.1. *The extremal algebra $Ea(J^2; unit)$ is isometrically isomorphic to the completion of $\mathbb{C}[\mathcal{I}_2]$ with respect to $\|\cdot\|$.*

When h is hermitian, so also is $a^{-1}ha$ whenever $\|a^{-1}\| = \|a\| = 1$. It follows that all words of odd length in u, v are hermitian; thus $x^n u$ is a hermitian involution for each $n \in \mathbb{Z}$.

Now let $k = \frac{1}{2}i(uv - vu) = \frac{1}{2}i(x - x^{-1})$. Then k is hermitian, and also $\frac{1}{2}i(x^2 - x^{-2})$ since u, vuv are hermitian. More generally, the element $k_n = \frac{1}{2}i(x^n - x^{-n})$ is hermitian for each $n \in \mathbb{N}$. Thus, $\mathbb{C}[\mathcal{I}_2]$ has a basis with "three quarters" of the elements being hermitian. In contrast, we shall see that each word of even length in $\mathcal{I}_2$, other than 1, has numerical range the closed unit disc $\mathbb{D}^-$.

PROPOSITION 2.2. *For all $n \in \mathbb{N}$,*

$$V(x^n) = V(x^{-n}) = V(\tfrac{1}{2}(x^n + x^{-n})) = \mathbb{D}^-.$$

Proof. Since $\|x\| = \|x^{-1}\| = 1$, we have $\mathrm{Sp}(x) \subseteq \mathbb{T}$. If, for $m \in \mathbb{N}$, we quotient out by $x^m = 1$ then, in the factor algebra, $\mathrm{Sp}(x)$ contains all m^{th} roots of unity. Hence, in $Ea(J^2; unit)$, $\mathrm{Sp}(x)$ contains all roots of unity and, since it is closed, $\mathrm{Sp}(x) = \mathbb{T}$. It follows that, for $n \in \mathbb{N}$, $\mathrm{Sp}(x^n) = \mathrm{Sp}(x^{-n}) = \mathbb{T}$ and hence that $V(x^n) = V(x^{-n}) = \mathbb{D}^-$, since the numerical range contains the convex hull of the spectrum.

Now let $n \in \mathbb{N}$ and quotient out by $x^{2n} = 1$ so that we are working with the extremal algebra over the finite dihedral group D_{2n}. To complete the proof it is enough to show that $V(x^n) = \mathbb{D}^-$ in this finite dimensional algebra, since $x^n = x^{-n}$. The norm is determined by the fact that all products of the exponential terms $\exp(i\tau u), \exp(i\tau v)$ $(\tau \in \mathbb{R})$ have norm 1. A straightforward induction argument shows that any such product w is of the form

$$w = \sum_{j=0}^{2n-1} \alpha_j x^j + i \sum_{j=0}^{2n-1} \beta_j x^j u,$$

where $\alpha_j, \beta_j \in \mathbb{R}$, and that $ww^* = 1$, where here * is the canonical involution on the group algebra $\mathbb{C}[D_{2n}]$. Equate coefficients of x^0 and x^n and we obtain

$$\sum_{j=0}^{2n-1} (\alpha_j^2 + \beta_j^2) = 1, \quad 2\sum_{j=0}^{n-1} (\alpha_j \alpha_{j+n} + \beta_j \beta_{j+n}) = 0.$$

It follows that

$$\sum_{j=0}^{n-1} [(\alpha_j \pm \alpha_{j+n})^2 + (\beta_j \pm \beta_{j+n})^2] = 1$$

and hence $|\alpha_0| + |\alpha_n| \leq 1$. Now define a functional on $\mathbb{C}[D_{2n}]$ by $f(1) = 1, f(x^n) = \zeta$ and $f(y) = 0$ for all other elements of D_{2n}. For $|\zeta| \leq 1$ we now get $|f(a)| \leq \|a\|$ for all a. This shows that $V(x^n)$ contains the closed unit disc. The reverse inclusions are obvious. □

Proposition 2.3. *For all* $n \in \mathbb{N}$,

$$V(k_n) = J \quad \textit{and} \quad V(k^{2n-1}) = J.$$

Proof. Since k is hermitian with

$$\mathrm{Sp}(k) = \{\tfrac{1}{2} i(z - z^{-1}) : z \in \mathbb{T}\} = J,$$

we have $V(k) = J$. This holds, similarly, for k_n $(n \in \mathbb{N})$. We also have

$$k^{2n-1} = (-4)^{1-n} \sum_{r=0}^{n-1} \binom{2n-1}{r} (-1)^r k_{2n-1-2r}$$

so that k^{2n-1} is hermitian for all $n \in \mathbb{N}$, with spectrum and numerical range equal to J. □

It is less obvious that, for each $n \in \mathbb{N}$, $V(k^{2n}) = \mathbb{H}^-$. This will be established later in Corollary 3.7. For the present, we have:

Lemma 2.4. *For all* $n \in \mathbb{N}$, $V(k^{2n}) \supseteq V(k^2) = \mathbb{H}^-$.

Proof. We have $k^2 = \frac{1}{2} - \frac{1}{4}(x^2 + x^{-2})$ and by Proposition 2.2, $V(\frac{1}{2}(x^2 + x^{-2})) = \mathbb{D}^-$. Hence $V(k^2) = \mathbb{H}^-$. Now quotient out $x^4 = 1$. Then, for $n \in \mathbb{N}$, $k^{2n} = k^2 = \frac{1}{2}(1 - x^2)$ and it follows from the proof of Proposition 2.2 that, in the factor algebra, $V(x^2) = \mathbb{D}^-$ and $V(k^{2n}) = \mathbb{H}^-$. □

3. The extremal algebra $Ea(J; odd)$. Let $Ea(J; odd)$ denote the extremal Banach algebra generated by h subject to the conditions $||h|| = 1$ and every odd positive power of h is hermitian. We shall show that $Ea(J; odd)$ can be identified with the closed subalgebra of $Ea(J^2; unit)$ generated by the element $k = \frac{1}{2}\,i(uv - vu)$.

Now let $h(t) = t$ $(t \in J)$. We define

$$\mathcal{P} = \{\textstyle\sum_{n=0}^{\infty}\alpha_n h^{2n+1} : \alpha_n \in \mathbb{R}, \sum|\alpha_n| < \infty\} \subseteq C[-1,1]$$

and then $\mathcal{G} = \exp(i\mathcal{P})$. For each $g = \exp(ip) \in \mathcal{G}$ and $t \in J$, we have $g(-t) = \overline{g(t)} = g^{-1}(t)$, $|g(t)| = 1$, $g(0) = 1$ and $g^{\alpha} = \exp(i\alpha p) \in \mathcal{G}$ $(\alpha \in \mathbb{R})$. We now define

$$\mathcal{A} = \{a = \textstyle\sum\alpha_n g_n : \alpha_n \in \mathbb{C}, g_n \in \mathcal{G}, \sum|\alpha_n| < \infty\}$$

and, for $a \in \mathcal{A}$, we define $||a||$ to be the infimum of $\sum|\alpha_n|$ over all such representations. The function algebra $\mathcal{A}$ is complete since $a_n \in \mathcal{A}$ with $\sum||a_n|| < \infty$ implies $\sum a_n \in \mathcal{A}$. As in [5], we have $h = \sum\alpha_n \exp(i\beta_n h)$ for certain $\alpha_n \in \mathbb{C}$ and $\beta_n \in \mathbb{R}$, with $\sum|\alpha_n| < \infty$, which shows that $h \in \mathcal{A}$. It is straightforward to verify that the polynomials are dense in $\mathcal{A}$.

Suppose that g is an element of a Banach algebra such that $||g|| = 1$ and every odd positive power of g is hermitian. Consider $b = \sum\beta_j g^j$ where $\beta_j \in \mathbb{C}$ with $\sum|\beta_j| < \infty$, and let $a = \sum\beta_j h^j$. Suppose that $a = \sum\alpha_n \exp(ia_n)$ where $\alpha_n \in \mathbb{C}$, $a_n = \sum_{j=1}^{\infty}\pi_{n,j}\, h^{2j-1}$, $\pi_{n,j} \in \mathbb{R}$, $\sum|\alpha_n| < \infty$ and $\sum_j|\pi_{n,j}| < \infty$ $(n \in \mathbb{N})$. Let $b_n = \sum_j \pi_{n,j}\, g^{2j-1}$. By [9] (the first Proposition on page 424), $b = \sum\alpha_n \exp(ib_n)$. Since b_n is hermitian, $||b|| \le \sum|\alpha_n|$. Taking the infimum over the expressions $\sum\alpha_n \exp(ia_n)$ for a we have $||b|| \le ||a||$. Thus we have proved:

THEOREM 3.1. *The extremal algebra $Ea(J; odd)$ is isometrically isomorphic to $(\mathcal{A}, ||\cdot||)$.*

Note that, as in the Wiener algebra, spectra of elements in $\mathcal{A}$ are given by evaluations in J (see, for example, [3]) so that $|\cdot|_\infty$ is the spectral radius in $\mathcal{A}$. For $a \in \mathcal{A}$ we define $a^*(t) = \overline{a(t)}$ $(t \in J)$ and we note that $a^* \in \mathcal{A}$. This agrees with the usual involution on $\ell^1(\mathcal{G})$, $\mathcal{G}$ being a group under pointwise multiplication.

LEMMA 3.2. *Let $a \in \mathcal{A}$ with $a = a^*$ and $a(-t) = -a(t)$ $(t \in J)$. Then $a \in \mathcal{P}$.*

Proof. Let $a = \sum\alpha_n g_n$. For $t \in J$, $a(t) = -a(-t) = -\sum\alpha_n g_n^{-1}(t)$ so that $a = -\sum\alpha_n g_n^{-1} = -(\sum\alpha_n g_n^{-1})^* = -\sum\overline{\alpha_n} g_n$. By replacing α_n with $\frac{1}{2}(\alpha_n - \overline{\alpha_n})$, we may assume that $\alpha_n \in i\mathbb{R}$. We have $a = \frac{1}{2}\sum\alpha_n(g_n - g_n^{-1})$. But $g = \exp(ip)$ with $p \in \mathcal{P}$ gives $g - g^{-1} = 2i\sin p \in i\mathcal{P}$. Therefore, $a \in \mathcal{P}$. □

LEMMA 3.3. *Let $a \in \mathcal{A}$ with $a(0) = 1$, $|a(t)| = 1$ and $a(-t) = \overline{a(t)}$ $(t \in J)$. Then $a \in \mathcal{G}$.*

Proof. Since a is continuous, there is a continuous odd real function b such that $a = \exp(ib)$. There is an odd real polynomial p such that $|b - p|_\infty < 1$. Let $g = \exp(ip)$. Then $|\exp(ib) - \exp(ip)|_\infty = 2|\sin\frac{1}{2}(p - b)|_\infty$ so that $|g^{-1}a - 1|_\infty < 1$. The analytic functional calculus now shows that $i\log(g^{-1}a) \in \mathcal{A}$ and satisfies the conditions of Lemma 3.2. Thus, $i\log(g^{-1}a) \in \mathcal{P}$ so that $g^{-1}a \in \mathcal{G}$ and $a \in \mathcal{G}$. □

LEMMA 3.4. *Let p be a real polynomial in ih with $p(0) \neq 0$. Then there exists $g \in \mathcal{G}$ such that $(pg)^* = pg$.*

Proof. We have $p(-t) = \overline{p(t)}$ $(t \in J)$. Any zeros of p in J are in pairs $\pm\alpha$. We may factor these out to assume, without loss, that p is never zero on J and hence $p^{-1} \in \mathcal{A}$. Then $q = p^*p^{-1}$ satisfies the conditions of Lemma 3.3. Hence $q \in \mathcal{G}$ so that $g = q^{1/2} \in \mathcal{G}$. Therefore, $p^*p^{-1} = g^2$ and $pg = p^*g^{-1} = (pg)^*$. □

PROPOSITION 3.5. *Let $p \in \mathcal{A}$ be a real polynomial in ih. Then $||p|| = r(p)$.*

Proof. Assume that $r(p) < 1$. We show that $||p|| \leq 1$. We have $p(t) = \sum \alpha_j (it)^j$ where $\alpha_j \in \mathbb{R}$ and $\alpha_0 \in (-1, 1)$. For $n \in \mathbb{N}$, define

$$p_n = p + (1 - \alpha_0)(1 - h^2)^n \quad \text{and} \quad q_n = p_n^* p_n,$$

so that q_n is a polynomial and $q_n(t) = |p_n(t)|^2$ $(t \in J)$. Then we have

$$q_n(0) = 1, \quad q_n'(0) = 0 \quad \text{and} \quad q_n''(0) = 2\alpha_1^2 - 4\alpha_2 - 4n(1 - \alpha_0).$$

Choose $n_0 \in \mathbb{N}$ such that $q_{n_0}''(0) < 0$, and $\delta \in (0, 1)$ such that $q_{n_0}(t) < 1$ $(0 < |t| < \delta)$. Therefore, if $0 < |t| < \delta$ then $|p_{n_0}(t)| < 1$; if also $m > n_0$ then $p_m(t)$ is a convex combination of $p(t)$ and $p_{n_0}(t)$; and if further $r(p) + 2(1 - \delta^2)^m < 1$ then we have $|p_m(t)| < 1$ $(0 < |t| \leq 1)$. Repeat the above for p replaced with $-p$ to give $\widetilde{p}_m = -p + (1 + \alpha_0)(1 - h^2)^m$ with the same properties — we may choose, and now fix, the same m by increasing exponents. Then $2p = (1 + \alpha_0)p_m - (1 - \alpha_0)\widetilde{p}_m$ and it is enough to show that $||p_m|| \leq 1$, since $||\widetilde{p}_m|| \leq 1$ follows similarly. By Lemma 3.4, there exists $g \in \mathcal{G}$ such that $q = p_m g$ satisfies $q^* = q$. Then $q^2 = q^*q = p_m^* p_m = q_m$. By the above properties of p_m, we have the factorisation $1 - q^2 = 1 - q_m = h^2 c$, where c is a polynomial in h with $c(t) > 0$ $(t \in J)$. By the analytic functional calculus, $\sqrt{c} \in \mathcal{A}$. By Lemma 3.3, $d = q + ih\sqrt{c}$ and $d^{-1} = q - ih\sqrt{c}$ are in $\mathcal{G}$. Hence, $||p_m|| = ||q|| = \frac{1}{2}||d + d^{-1}|| \leq 1$. □

As before we write $x = uv$ and $k = \frac{1}{2}i(x - x^{-1})$. Recall that $||k|| = 1$ and all odd positive powers of k are hermitian. Since $Ea(J; odd)$ is the extremal algebra subject to this condition, it is immediate that $||P(k)|| \leq ||P(h)||$ for all polynomials P. In fact, we have:

THEOREM 3.6. *The extremal algebra $Ea(J; odd)$ is isometrically isomorphic to the subalgebra of $Ea(J^2; unit)$ generated by the element $\frac{1}{2}i(uv - vu)$, where u, v are the generators of $Ea(J^2; unit)$.*

Proof. Let $\mathcal{H}$ be the subgroup of $\mathcal{I}_2$ generated by x, and let $\mathcal{B} = \mathbb{C}[\mathcal{H}]^-$ so that $\mathcal{B}$ is a commutative subalgebra of $Ea(J^2; unit) = \mathcal{K}$ (say). Define a homomorphism $\Phi : \mathcal{A} \to \mathcal{B}$ by $\Phi(h) = k$. Let $\mathcal{C} = \Phi(\mathcal{A})$. Since $\mathrm{Sp}(k) = J$, Φ is injective. Hence, for $c \in \mathcal{C}$, we may define $|c| = ||a||$, where $\Phi(a) = c$. We show that $|c| = ||c||$ $(c \in \mathcal{C})$.

We have $||\Phi(a)|| \leq ||a||$, and so $||c|| \leq |c|$. Let $a_0 \in \mathcal{C}$. Then there exists $\phi \in \mathcal{C}'$ such that $|\phi| = 1$ and $\phi(a_0) = |a_0|$. We show that there exists an extension $\widetilde{\phi} \in \mathcal{K}'$ such that $||\widetilde{\phi}|| \leq 1$. Then $||a_0|| \geq \widetilde{\phi}(a_0) = |a_0|$.

Consider $a \in \mathcal{J}$. By induction, $a^*a = 1$ and $a = b + icu$ with $b, c \in \mathbb{R}[\mathcal{H}]$. For all $d \in \mathbb{R}[\mathcal{H}]$, we have $ud = d^*u$ which gives $1 = a^*a = b^*b + c^*c$. Let $\chi \in \mathcal{B}'$ be a multiplicative linear functional. Since $\chi(d^*) = \overline{\chi(d)}$ $(d \in \mathcal{B})$, we have $|\chi(b)|^2 + |\chi(c)|^2 = 1$. Hence $|\chi(b)| \leq 1$ and so $r(b) \leq 1$. Define an isomorphism of period 2 on $\mathbb{C}[\mathcal{I}_2]$ by $(x^j u^m)^\dagger = (-x)^{-j} u^m$ $(j \in \mathbb{Z},\ m = 0, 1)$. Then $\mathcal{J}^\dagger = \mathcal{J}$ and so the isomorphism extends to an isometric isomorphism of $\mathcal{K}$. We have $k^\dagger = k$ and so $c^\dagger = c$ $(c \in \mathcal{C})$. For $j \in \mathbb{Z}$, $x^j + (-x)^{-j}$ is a real polynomial in $x - x^{-1} = 2ik$. Hence, if $d \in \mathbb{C}[\mathcal{H}]$ (respectively $\mathbb{R}[\mathcal{H}]$) then $d + d^\dagger$ is a polynomial (respectively real polynomial) in ik. Let $p = \frac{1}{2}(b + b^\dagger)$. We have $r(b^\dagger) = r(b) \leq 1$ and so $r(p) \leq 1$. By the above, $p = \sum \pi_j (ik)^j$ where $\pi_j \in \mathbb{R}$. Let $q = \sum \pi_j (ih)^j$. Since $\mathrm{Sp}(k) = \mathrm{Sp}(h)$, $r(q) = r(p) \leq 1$. By Proposition 3.5, $||q|| \leq 1$. Hence $|p| = |\Phi(q)| = ||q|| \leq 1$. Therefore $|\phi(p)| \leq |\phi|\,|p| \leq 1$.

Any $a \in \mathbb{C}[\mathcal{I}_2]$ can be written uniquely $a = b + icu$ with $b, c \in \mathbb{C}[\mathcal{H}]$. Then define $\psi(a) = \frac{1}{2}\phi(b + b^\dagger)$. The above shows that if $a \in \mathcal{J}$ then $|\psi(a)| \leq 1$. Hence $||\psi|| \leq 1$. If $d \in \mathcal{C} \cap \mathbb{C}[\mathcal{I}_2]$ then $d^\dagger = d$ and $\psi(d) = \phi(d)$. Since the polynomials are dense in $\mathcal{A}$, $\mathcal{C} \cap \mathbb{C}[\mathcal{I}_2]$ is $|\cdot|$-dense and hence $||\cdot||$-dense in $\mathcal{C}$. So ψ may be extended to the required $\widetilde{\phi}$. □

COROLLARY 3.7. *Let $k = \frac{1}{2}i(uv - vu)$ where u, v are the generators of $Ea(J^2; unit)$. Then for $n \in \mathbb{N}$, $||k^{2n} - \frac{1}{2}|| = \frac{1}{2}$ and $V(k^{2n}) = \mathbb{H}^-$.*

Proof. Define $a \in \mathcal{A}$ by $a(t) = 1 - 2t^{2n}$. Then a satisfies the conditions of Proposition 3.5 so that $||a|| \leq 1$. Hence, by Theorem 3.6, $||1 - 2k^{2n}|| \leq 1$ so that $V(k^{2n}) \subseteq \mathbb{H}^-$, and applying Lemma 2.4 completes the proof. □

4. The extremal algebra *Ea(J; even)*. Let $Ea(J; even)$ denote the extremal Banach algebra generated by h (not hermitian) subject to the conditions $||h|| = 1$ and every even positive power of h is hermitian. The Vidav-Palmer theorem shows that the subalgebra generated by 1 and h^2 is a commutative unital monogenic C^*-algebra, and so is the algebra of continuous functions on $\mathrm{Sp}(h^2)$. Certainly this spectrum is contained in J so that $\mathrm{Sp}(h)$ is contained in $[-1, 1] \cup [-i, i]$. We shall see that, in fact, equality holds in both cases. The extremal algebra may be realised as follows.

Let $K = [-1, 1] \cup [-i, i]$ and let $\mathcal{E}$ be all continuous $f : K \to \mathbb{C}$ such that

$$\widetilde{f}(0) = \lim_{t \to 0} \frac{f(t) - f(-t)}{2t}$$

exists. Then $\mathcal{E}$ is an algebra under pointwise operations. For $f \in \mathcal{E}$ and $t \in K$, define

$$f_1(t) = \frac{f(t) + f(-t)}{2}, \qquad f_2(t) = \begin{cases} \dfrac{f(t) - f(-t)}{2t} & \text{if } t \neq 0, \\ \widetilde{f}(0) & \text{if } t = 0. \end{cases}$$

For $f \in \mathcal{E}$, f_1 and f_2 are the unique even continuous functions on K such that

$$f(t) = f_1(t) + t f_2(t) \quad (t \in K).$$

Conversely, $f \in \mathcal{E}$ whenever f has such a decomposition. For $f \in \mathcal{E}$, define

$$||f|| = |f_1|_\infty + |f_2|_\infty.$$

Then $||\cdot||$ is an algebra norm and $(\mathcal{E}, ||\cdot||)$ is a Banach algebra since, as a normed space, it is the ℓ^1 direct sum of two continuous function spaces. Since the even polynomials are $|\cdot|_\infty$-dense in the even continuous functions it follows that the polynomials are $||\cdot||$-dense in $(\mathcal{E}, ||\cdot||)$.

Now let h be the identity function on K. Then $h \in \mathcal{E}$ with $||h|| = 1$. It follows directly from the definition of spectrum that $\mathrm{Sp}(h) = K$ so that all even positive powers of h have spectrum J.

PROPOSITION 4.1. *For $n \in \mathbb{N}$,*

$$V(h^n) = J \ (n \text{ even}) \quad \text{and} \quad V(h^n) = \mathbb{D}^- \ (n \text{ odd}).$$

Generalising the odd power case, for any even function $f \in \mathcal{E}$, $V(hf)$ is the closed disc, centre 0 and radius ρ, where $\rho = |f|_\infty$.

Proof. For $n \in \mathbb{N}$, n even, $||\exp(i\tau h^n)|| = 1$ $(\tau \in \mathbb{R})$, so that h^n is hermitian, and hence $V(h^n) = J$, the convex hull of the spectrum.

For $f \in \mathcal{E}$, f an even function, and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, we have

$$\sup \mathrm{Re}\, V(\lambda hf) = \lim_{\alpha \to 0+} \frac{||1 + \alpha\lambda hf|| - 1}{\alpha} = \lim_{\alpha \to 0+} \frac{(1 + \alpha\rho) - 1}{\alpha} = \rho,$$

giving $V(hf)$ as stated. □

Finally, we have the following identification.

THEOREM 4.2. *The extremal algebra $Ea(J; even)$ is isometrically isomorphic to $(\mathcal{E}, ||\cdot||)$.*

Proof. Let $h(t) = t$ $(t \in K)$. Then $|h| = 1$ and all even positive powers of h are hermitian. To see that $||\cdot||$ is extremal, let g be any Banach algebra element with $||g|| = 1$ and all even powers of g hermitian, let P be any polynomial and let Q and R be the even polynomials such that, for all t, $P(t) = Q(t) + tR(t)$. Then, since $Q(g)$ and $R(g)$ are in the algebra generated by 1 and g^2, and $\mathrm{Sp}(g) \subseteq K = \mathrm{Sp}(h)$, we have

$$||P(g)|| \le ||Q(g)|| + ||R(g)|| = r(Q(g)) + r(R(g)) \le |Q(h)|_\infty + |R(h)|_\infty = ||P(h)||$$

and the result follows. □

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