Some Remarks Concerning Polynomial Approximation in the Mean

by

John Akeroyd and Elias G. Saleeby

UofA-R-168

Some Remarks Concerning Polynomial Approximation in the Mean

John Akeroyd and Elias G. Saleeby

For any finite, positive Borel measure μ with compact support in the complex plane $\mathbf{C}$, J. Thomson has given (cf. [Th1]) a direct sum decomposition of $P^t(d\mu)$ (the closure of the polynomials in $L^t(d\mu)$; $1 \leq t < \infty$) that involves the components of the analytic bounded point evaluations of $P^t(d\mu)$. To clarify our terminology, a point z in $\mathbf{C}$ is called an *analytic bounded point evaluation* for $P^t(d\mu)$ if there are positive constants M and r such that $|p(w)| \leq M\|p\|_{L^t(d\mu)}$ for all polynomials p and all w such that $|z-w| < r$; the set of all points z of this type is denoted $abpe(P^t(d\mu))$. This far-reaching result of J. Thomson has numerous ramifications in both function theory and operator theory, yet its proof (which involves establishing a dichotomy using Cauchy transforms of measures) does not lend itself to the determination of $abpe(P^t(d\mu))$ for a particular measure μ. There are just a handful of results in the literature that give analytic conditions that help determine $abpe(P^t(d\mu))$ for certain measures μ; the most prominent of these is Szegö's Theorem.

Szegö's Theorem. Let μ be a finite, positive Borel measure with support in $\{z : |z| = 1\}$ and let $d\mu = hdm + d\mu_s$ be the Lebesgue Decomposition of μ with respect to m (normalized Lesbesgue measure on $\{z : |z| = 1\}$); $\mu_s \perp m$. Then,

$$\inf\{\int |\frac{1}{z} - p|^t d\mu : p \text{ is a polynomial}\} = \exp(\int \log(h) dm) \text{ for } 0 < t < \infty.$$

Corollary 1. Let μ and m be as in Szegö's Theorem.

1) If $\int \log(h)dm > -\infty$, then $abpe(P^t(d\mu)) = \mathbf{D}$ $(1 \le t < \infty)$.

2) If $\int \log(h)dm = -\infty$, then $abpe(P^t(d\mu)) = \emptyset$ $(1 \le t < \infty)$.

There is a short list of extensions of Szegö's Theorem ([V], 1978; [Tr], 1979; [A1], 1994; [A2], 1997) – each truely extending the contemporary status of the theory – yet none of these provide a Szegö–type condition that would determine (for example) whether or not $\frac{1}{z} \in P^t(d\mu)$ for general measures μ with support in $\{z : \frac{1}{2} \le |z| \le 1\}$. Is such a condition attainable? In [A1] and [A2] the first author briefly examined this question as part of a general discussion of Szegö's Theorem. In this note, we take another look at this question in light of some recent examples. We finish with an example of a measure μ for which $abpe(P^t(d\mu))$ varies continuously with t.

Proposition 2. Let $\{z_n\}$ be a Blaschke sequence in $\mathbf{D} := \{z : |z| < 1\}$ and let $\{c_n\}$ be a summable sequence of positive real numbers. Let ν be a finite positive Borel measure with support in $\partial\mathbf{D}$ such that $\nu \ll m$ $(d\nu = hdm)$ and let $\mu = \nu + \sigma$, where $\sigma = \Sigma_{n=1}^{\infty} c_n \delta_{z_n}$.

(1) If $\int \log(h)dm > -\infty$, then $abpe(P^t(d\mu)) = \mathbf{D}$ $(1 \le t < \infty)$.

(2) If $\int \log(h)dm = -\infty$, then $abpe(P^t(d\mu)) = \emptyset$ $(1 \le t < \infty)$.

This proposition tells us that the outcome of Szegö's Theorem for a measure with support in $\partial\mathbf{D}$ cannot be effected by adding to that measure a summable series of weighted point masses evaluated at the points of a Blaschke sequence.

Proof. (1) Since $\int \log(h)dm > -\infty$, it follows from Szegö's Theorem that $abpe(P^t(d\nu)) = \mathbf{D}$ $(1 \leq t < \infty)$. Since $\mu = \nu + \sigma$ (whose support is contained in $\overline{\mathbf{D}}$), we have that $\mathbf{D} \subseteq abpe(P^t(d\mu)) \subseteq \mathbf{D}$, and so (1) holds.

(2) By a Möbius transformation of $\mathbf{D}$, we may assume that $z_n \neq 0$ for all n. Now since $\int \log(h)dm = -\infty$, by our corollary to Szegö's Theorem and [AS] (Lemma 2.1), 0 is not a bounded point evaluation for $P^t(d\nu)$. So there is a sequence of polynomials $\{p_n\}$ such that $\frac{|p_n(0)|}{||p_n||_{L^t(d\nu)}} \to \infty$ as $n \to \infty$. If B is the Blaschke product whose associated sequence is $\{z_n\}$, then $Bp_n \in P^t(d\mu)$ for all n and

$$\frac{|B(0)p_n(0)|}{||Bp_n||_{L^t(d\mu)}} = |B(0)| \cdot \left(\frac{|p_n(0)|}{||p_n||_{L^t(d\nu)}}\right) \to \infty \text{ as } n \to \infty.$$

So 0 is not a bounded point evaluation for $P^t(d\mu)$ and, by [AS] (Lemma 2.1), $abpe(P^t(d\mu)) = \emptyset$.□

Let us return to the hypothesis of our proposition. Do we need the restricition that the points $\{z_n\}$ (in $\mathbf{D}$) form a Blaschke sequence? The answer is a definite yes. In fact, in one of the simplest imaginable cases – $z_n = 1 - \frac{1}{n}$ for $n = 1, 2, 3, \ldots$ – there is a measure ν on $\partial\mathbf{D}$ $(d\nu = hdm)$ such that $\int_{\partial\mathbf{D}} \log(h)dm = -\infty$ and yet, for t sufficiently large, $abpe(P^t(d\mu)) = \mathbf{D}$; $\mu := \nu + \sum_{n=1}^{\infty} \frac{1}{n^2} \cdot \delta_{z_n}$. The first example of such a measure μ was given by T. Kriete (see [K]). Recently (see [AS]) the authors produced a whole class of measures μ of this type and examined $abpe(P^t(d\mu))$ for $1 \leq t < \infty$. These measures μ have the property that $abpe(P^t(d\mu))$ varies with t; by Szegö's Theorem, such an event is impossible if support$(\mu) \subseteq \partial\mathbf{D}$. As an aside, it should be noted that, to this point, only measures that are generated by the Residue Theorem applied to non-Nevanlinna class function have broken away from the rather monotone result of Szegö's Theorem (see [AS] and [Th2]). The existence of such measures, though, indicates that a Szegö–type theorem for measures with suport in $\{z : \frac{1}{2} \leq z \leq 1\}$ would be difficult to obtain. The analytic condition for such measures (unlike that given by Szegö's Theorem) would have to depend upon t and for a fixed t would have to be sensitive to even a mild adjustment in a given measure (e.g., a very slight change in the spacing of point masses – see [AS]). Furthermore, the condition (again, unlike that given by Szegö's Theorem) would also have to be sensitive to changes in any symmetry of a given measure. To be precise, if $\nu \ll m$ ($d\nu = hdm$) and $h(e^{-i\theta}) = h(e^{i\theta})$ for $0 \leq \theta \leq \pi$, then, by Szegö's Theorem, $abpe(P^t(d\nu)) = \mathbf{D}$ if and only if $abpe(P^t(d\nu^*)) = \mathbf{D}$ where $\nu^* := \nu|_I + m|_J$; $I := \{z \in \partial\mathbf{D} : \mathrm{Im}(z) \geq 0\}$ and $J := \{z \in \partial\mathbf{D} : \mathrm{Im}(z) \leq 0\}$. If, however, the support of ν is $\partial\mathbf{D}$ along with a (non–Blaschke) sequence of points in $\mathbf{D}$, then such a change in the symmetry of ν on $\partial\mathbf{D}$ can dramatically effect the set of analytic bounded point evaluations (see [AS], Example 3.5).

In our discussion above, the points $\{z_n\}$ lie on a radial segment in $\mathbf{D}$. If these points were on a nontangential segment, the outcome would be similar. However, if these points (in $\mathbf{D}$) were to approach $\partial\mathbf{D}$ tangentially, then the result changes. In particular, consider $\Gamma := \{z : |z - \frac{1}{2}| = \frac{1}{2}\}$, and let $\Gamma^+ = \{z \in \Gamma : \mathrm{Im}(z) \geq 0\}$ and $\Gamma^- = \{z \in \Gamma : \mathrm{Im}(z) \leq 0\}$. Define $\{z_n\}$ in $\mathbf{D}$ by:

$$z_1 = 0,\ z_{2n} = \Gamma^+ \cap \{z : |z-1| = \tfrac{1}{n}\} \text{ and}$$
$$z_{2n+1} = \Gamma^- \cap \{z : |z-1| = \tfrac{1}{n}\} \text{ (for } n = 1,2,3,...).$$

Then $\{z_n\}$ approaches 1 (in $\mathbf{D}$) at the same rate as in the radial case, but here $\{z_n\}$ forms a Blaschke sequence and so Proposition 2 applies.

Question 3. Does there exist a sequence $\{z_n\}$ in $\Gamma\backslash\{1\}$ and a summable sequence $\{c_n\}$ of positive constants such that if $d\nu(z) := e^{-\frac{1}{|1-z|}}dm(z)$, $\sigma := \sum_{n=1}^{\infty} c_n\delta_{z_n}$ and $\mu = \nu + \sigma$, then $abpe(P^1(d\mu)) = \mathbf{D}$?

We observe that if $\{z_n\} \subseteq \Gamma\backslash\{1\}$ and either $\{z_n\} \cap \Gamma^+$ is a Blaschke sequence or $\{z_n\} \cap \Gamma^-$ is a Blaschke sequence, then by the proof of Proposition 2 and by [A2] Theorem 1, $abpe(P^t(d\mu)) = \emptyset$ for $1 \le t < \infty$.

We conclude this note with an example of a measure μ for which the $abpe(P^t(d\mu))$ varies continously with t. We first state a rather well-known preliminary result.

Lemma 4. Let μ be a finite, positive Borel measure with compact support K in $\mathbf{C}$. Let $\{\Delta_n\}_{n=1}^{N}$ (N may be infinity) be a Borel partition of K. Then $P^t(d\mu) = \bigoplus_{n=1}^{N} P^t(d\mu|_{\Delta_n})$ if and only if $\chi_{\Delta_n} \in P^t(d\mu)$, $1 \le n < N$, $(1 \le t < \infty)$.

Now, let r_k be an enumeration of $\mathbf{Q} \cap (1,\infty)$, and let $\rho_k = \frac{1}{2^k}$ $(k = 1,2,3,...)$. Define a collection of mutually disjoint discs $\{D_k\}_{k=1}^{\infty}$ having radii ρ_k and centered at x_k, where $x_1 = 0$, $x_2 = 1$, $x_3 = 1 + \frac{1}{2}$, $x_4 = 1 + \frac{1}{2} + \frac{1}{4}$, and so on; in general, $x_k = 1 + \sum_{n=1}^{k-2} \rho_n$ for $k \ge 3$. Note that by construction, $dist(D_k, D_{k+1}) = \rho_{k+1}$ for $k \ge 1$, and $x_k \to 2$ as $k \to \infty$. Now, let $K = \left(\cup_k \overline{D}_k\right) \cup \{2\}$, and note that both K and $K\backslash\overline{D}_k$ are compact sets. Define a measure μ^*_{1,r_k} with support contained in $\overline{D}_k$ for each $k \ge 1$, by $\mu^*_{1,r_k} := \frac{1}{2^k}(\mu_{1,r_k} \circ \psi_k^{-1})$, where ψ_k is the analytic linear transformation given by $\psi_k(z) = \rho_k z + x_k$, $(k = 1,2,3,...)$. Observe that $abpe(P^t(d\mu^*_{1,r_k})) = \emptyset$ if $t < r_k$ and $abpe(P^t(d\mu^*_{1,r_k})) = D_k$ if $t \ge r_k$ (by Theorem 3.2 in [AS]). Let μ be the finite, positive Borel measure with support K given by $\mu = \sum_{k=1}^{\infty} \mu^*_{k,r_k}$.

Proposition 5. With μ as defined above, $abpe(P^t(d\mu))$ varies continuously with t, $1 \leq t < \infty$.

Proof. Select k in $\mathbb{N}$ and choose Jordan regions V and W such that: (1) $\overline{V} \cap \overline{W} = \emptyset$, (2) $\overline{D}_k \subseteq V$ and (3) $K \backslash \overline{D}_k \subseteq W$. Define f on $V \cup W$ by $f \equiv 1$ on V and $f \equiv 0$ on W. Since f is analytic in a neighborhood of K, we can apply Runge's Theorem and find a sequence of polynomials that converges uniformly to f. Hence, $\chi_{\overline{D}_k} \in P^t(d\mu)$, and thus, by Lemma 4, $P^t(d\mu) = \bigoplus_{n=1}^{\infty} P^t(d\mu^*_{1,r_n})$ for $1 \leq t < \infty$. Since $\mathbb{Q} \cap (1, \infty)$ is dense in $[1, \infty)$, we can now conclude that $abpe(P^t(d\mu))$ varies continuously with t for $1 \leq t < \infty$.□

References

[A1] J. Akeroyd, *An extension of Szegö's Theorem,* Indiana Univ. Math. J., Vol. 43 (1994) 1339-1347.

[A2] J. Akeroyd, *An extension of Szegö's Theorem II,* Indiana Univ. Math. J., Vol. 45 (1996), 241-252.

[AS] J. Akeroyd and E.G. Saleeby, *A class of $P^t(d\mu)$ spaces whose point evaluations vary with* t, Proc. Amer. Math. Soc., (to appear).

[K] T. L. Kriete, *Cosubnormal dilation semigroups on Bergman spaces*, J. Operator Theory 17 (1987), 191-200.

[Th1] J. Thomson, *Approximation in the mean by polynomials*, Ann. of Math. (2) 133 (1991), 477-507.

[Th2] J. Thomson, *Bounded point evaluations and polynomial approximation*, Proc. Amer. Math. Soc.,Vol. 123 (1995), 1757-1761.

[Tr] T. Trent, *Extension of a theorem of Szegö*, Mich. Math. J. 26 (1979), 373-377.

[V] A. Vol'berg, *Mean square completeness of polynomials beyond the scope of Szegö's Theorem*, Dokl. Akad. Nauk. U.S.S.R., 241 (1978), 521-527, English Transl. Sov. Math. Dokl. 19 (877-881).

Address:
Department of Mathematical Sciences
301 Science and Engineering Building
University of Arkansas
Fayetteville, Arkansas, 72701
U.S.A.

Email addresses: jakeroyd@comp.uark.edu esaleeby@comp.uark.edu

Department of Mathematical Sciences
University of Arkansas, Research Reports

No. 154	Quasideterminants, I	I. Gelfand and V. Retakh	Oct. '97
No. 155	Nonlinear Carleman Operators on Banach Lattices	William Feldman	Oct. '97
No. 156	Piecewise-Smooth Surfaces as the Union of Geodesic Disks	Chaim Goodman-Strauss	Oct. '97
No. 157	On A Generalized Reflection Law For Functions Satisfying The Helmholtz Equation	Dawit Aberra	Nov. '97
No. 158	The Dual of Bergman Metric VMO	Daniel H. Luecking	Jan. '98
No. 159	Bounded Composition Operators with Closed Range on the Dirichlet Space	Daniel H. Luecking	Jan. '98
No. 160	An aperiodic pair of tiles in E^n for all $n \geq 3$	Chaim Goodman-Strauss	Feb. '98
No. 161	Projectively equivalent metrics, exact transverse line fields and the geodesic flow on the ellipsoid	Serge Tabachnikov	Mar. '98
No. 162	Best Approximation in the Mean by Analytic and Harmonic Functions	Dmitry Khavinson, John E. McCarthy, and Harold S. Shapiro	May '98
No. 163	Plemelj Projection Operators over Domain Manifolds	John Ryan	May '98
No. 164	Fagnano orbits of polygonal dual billiards	Serge Tabachnikov	May '98
No. 165	Monotonicity principles for singular integral equations in Clifford analysis	Swanhild Bernstein	May '98
No. 166	Sampling and the Closure of the Polynomials in a Weighted Hardy Space	John Akeryod and Elias G. Saleeby	June '98
No. 167	Nonlinear singular integral equations involving the Hilbert transformation in Clifford analysis	Swanhild Bernstein	July '98
No. 168	Some Remarks Concerning Polynomial Approximation in the Mean	John Akeroyd and Elias	July '98