The Conformal Laplacian on Spheres and Hyperbolas via Clifford Analysis

by

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Abstract. Using Clifford analysis we shall present a factorization of the Laplacian acting on functions defined on domains on spheres. This factorization leads us to investigate properties of a first order differential operator on the sphere, and to determine a Cauchy integral formula for functions annihilated by this operator. We determine a conformal covariance for both the operator and the integral over the sphere and we pull back the operator and representaion formula via a Cayley transformation to obtain similar results on R^{n-1}. The differential operator that we use is different from the spherical Dirac operator introduced by Cnops and Malonek. We use the integral formula to construct a Poisson formula markedly like the one for upper half space. We use this Poisson formula to solve the Dirichlet problems on a hemisphere. Furthermore we show that the factorization gives rise to two seperate Green's type formulas for solutions to our conformal Laplace equation.

Dedicated to Richard Delanghe on the occassion of his 60th birthday.

1. Introduction

In [8, 9]and elsewhere Cayley transformations are used to show that a valid analogue of Cauchy's integral formula and a corresponding Dirac operator may be constructed over the sphere and over the hyperbola. These in turn are used to develop a suitable version of Clifford analysis in this context. Independantly Cnops and Malonek [3] use Stokes' theorem and an extension argument from the sphere to R^n to give an explicit representation for the spherical Dirac operator. Furthermore Van Lanker in [13, 14, 15] uses Gagenbauer functions to also develop a function theory for this operator and related Dirac operators on the sphere.

While this analysis opens the door to many interesting results it is also natural to ask if there is also a suitable analogue to the Laplacian on the sphere and on the hyperbola and if one can introduce and study interesting boundary value problems over domains on the sphere and on the hyperbola. This question was answered in the affirmative in

[4] and [5]. In those works a suitable analogue of the Laplacian was found together with other higher order operators. It is shown that these operators are conformally equivalent to the operators D^k in euclidean space, where D is the euclidean Dirac operator and k is an arbitrary positive integer. When $k = 2$ this operator becomes the Laplacian in euclidean space. For the spherical Laplacian a Green's formula is introduced in [4, 5] and this formula bears a striking resemblance to the standard Green's formula in euclidean space.

Here we continue this analysis. We show that the factorization of the spherical Laplacian in terms of Dirac operators leads to two different versions of the spherical Green's formula. In turn this idea leads to a number of inequivalent Cauchy type formulas for the spherical and hyperbolic analogue of the operator D^k for k greater than one. We introduce the fundamental solution to a principle Dirac operator in this family and show that this to, regarded as an integral operator, is conformally invariant on the sphere. We also pull back this kernel and Dirac operator via the Cayley transformation and construct further operators and Cauchy type integral formulas on $\mathbf{R}^{n-1}$

Our analysis also leads to a Poisson kernel on the hemisphere. This kernel bears a marked resemblance to the classical Poisson kernel for upper half space. We use the Poisson kernel introduced here to solve the Dirichlet problems with L^p data for $1 < p < \infty$ on the boundaries of the hemisphere.

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2. Preliminaries

Here we will introduce the background material that we need to develop our main results. We begin by considering the real Clifford algebra Cl_n generated from $\mathbf{R}^n$ equipped with a negative definite inner product. So we assume that $\mathbf{R}^n \subset Cl_n$. If $e_1, \ldots, e_n$ is an orthonormal basis for $\mathbf{R}^n$ then $e_ie_j + e_je_i = -2\delta_{ij}$. For most of the algebraic details on this algebra we refer to [7]. An important group lying within Cl_n is the Pin group, which is defined to be $\{a \in Cl_n : a = a_1 \ldots a_p : p \in \mathbf{N}$ and $a_j \in S^{n-1}$ for $1 \leq j \leq p\}$. This group is denoted by $Pin(n)$. There is an antiautomorphism

$$\sim: Cl_n \to Cl_n :\sim (e_{j_1} \ldots e_{j_r}) = e_{j_r} \ldots e_{j_1}.$$

It is usual to write $\tilde{a}$ instead of $\sim a$ for $a \in Cl_n$.

DEFINITION 1. *Suppose U is a domain in $\mathbf{R}^{n-1}$, the span of $e_1, \ldots e_{n-1}$, then a differentiable function $f : U \rightarrow Cl_{n-1}$, is said to be left monogenic if $\Sigma_{j=1}^{n-1} e_j \frac{\partial f(x)}{\partial x_j} = 0$ for each $x \in U$ where Cl_{n-1} is the Clifford subalgebra of Cl_n generated from $\mathbf{R}^{n-1}$,*

Similarly a differentiable function $g : U \rightarrow Cl_{n-1}$ is said to be right monogenic if $\Sigma_{j=1}^{n-1} \frac{\partial g(x)}{\partial x_j} e_j = 0$ for each $x \in U$. We usually denote the differential operator $\Sigma_{j=1}^{n-1} e_j \frac{\partial}{\partial x_j}$ by D. So the left monogenic function f satisfies the equation $Df = 0$ on U and the right monogenic function g satisfies the equation $gD = 0$ on U. It should be noted that a function f is left monogenic if and only if $\tilde{f}$ is right monogenic. Also the function $G(x) = \frac{x}{\|x\|^{n-1}}$ is an example of a function which is both left and right monogenic. A simple but important point to note is that $D^2 = -\triangle_{n-1}$ which is the negative of the Laplacian in R^{n-1}.

Following [2] and elsewhere one can determine that if f and g are respectively left and right monogenic functions defined on a domain U and M is a compact, piecewise smooth hypersurface lying in U and bounding a subdomain of U then $\int_M g(x)n(x)f(x)d\sigma(x) = 0$ where $n(x)$ is the outward pointing unit vector at $x \in M$ and normal to M at x and σ is the usual Lebesgue measure for M. Of course this integral formula is a generalization of Cauchy's Theorem.

Suppose now that ψ is a Möbius transformation acting over $\mathbf{R}^{n-1} \cup \{\infty\}$ then in [1] and elsewhere it is shown that there are elements a, b, c and d of Cl_{n-1} such that $\psi(x) = ax + b)(cx + d)^{-1}$. The Clifford numbers a, b, c and d satisfy certain constraints specified in [1].

Using the previously mentioned Cauchy Theorem one can show, see [11], that if $f(y)$ is left monogenic and $y = \psi(x)$ then the function $J(\psi, x)f(\psi(x))$ is left monogenic in the vector variable x, where $J(\psi, x) = \frac{\widetilde{(cx+d)}}{\|cx+d\|^{n-1}}$. In fact this result primarily follows by noting that under a Möbius transformation the volume element $n(y)d\sigma(y)$ transforms to $\tilde{J}(\psi, x)n(x)J(\psi, x)d\sigma(x)$. See for instance [11] for details.

In Clifford analysis the analogue of Cauchy's integral formula is given by

$$f(y) = \frac{1}{\omega_{n-1}} \int_M G(x - y)n(x)f(x)d\sigma(x)$$

where y is a point in the domain bounded by the hypersurface M and ω_{n-1} is the surface area of the unit sphere in $\mathbf{R}^{n-1}$. Now if $\psi(x) = u$ and $\psi(y) = v$ then it may be determined that

$$G(u - v) = J(\psi, y)^{-1} G(x - y) J(\psi, x)^{-1}.$$

It follows that Cauchy's integral formula remains valid under Möbius transformations.

In [8, 9] we noted that instead of considering Möbius transformations acting over $\mathbf{R}^{n-1} \cup \{\infty\}$ we could consider the Cayley transformation $C(x) = (e_n x + 1)(x + e_n)^{-1}$ from $\mathbf{R}^{n-1}$ to the unit sphere, S^{n-1}, lying in $\mathbf{R}^n$. In [8, 9] we are able to use this transformation to carry over much of basic Clifford analysis to the setting of the sphere. We also used another Cayley transformation to carry over the same type of analysis to the $(n-1)$-dimensional hyperbola. We shall say more on that later. Using Stokes' Theorem and the invariance of monogenicity under Möbius transformations, we were able to infer the existene of a Dirac operator D_S with Cauchy kernel $G'(x-y) = \frac{x-y}{\|x-y\|^{n-1}}$. Here x and $y \in S^{n-1}$. Independandtly Cnops and Malonek [3] used an extension argument from the sphere to $\mathbf{R}^n$ to show that $D_S = x(\Gamma - \frac{n-1}{2})$, where $x \in S^{n-1}$ and Γ is the restriction to S^{n-1} of $x \wedge D_n$. Furthermore $\wedge$ is the usual vector wedge product and D_n is the Dirac operator in $\mathbf{R}^n$.

Using the fact that for each pair x and $y \in S^{n-1}$ then $\|x-y\|^2 = 2 - 2 < x, y >$ where $<,>$, denotes the usual inner product in $\mathbf{R}^n$, and $x\Gamma < x, y > = x \wedge y$, and $\Gamma x f(x) = -x\Gamma f(x) + (n-1)xf(x)$ for each Cl_n valued C^∞ function f defined on a domain on S^{n-1}, one may show directly that $D_S G' = G' D_S = 0$.

The identity $\Gamma x f(x) = -x\Gamma f(x) + (n-1)xf(x)$ may be easliy determined via direct calculation but may also be motivated by the fact shown in [12] that if f is a Cl_n valued, real analytic function defined on S^{n-1} then f is the restriction to S^{n-1} of a left monogenic function F. While the restriction of the left monogenic function $\frac{x}{\|x\|^n}F$ is xf, and the diference between eigenvalues for the operator Γ acting on the functions f and xf is $(n-1)$.

3. The Conformal Laplacian

We now turn to introduce a second order differential operator on the sphere that would play the same role in this context as the Laplacian in Euclidean space. If we consider the fundamental solution $\frac{1}{\|u-v\|^{n-3}}$ to the Laplacian in euclidean space it may be observed that the convolution

$$\int_{R^{n-1}} \frac{1}{\|u-v\|^{n-3}} h(u) du^{n-1}$$

conformally transforms to

$$\int_{S^{n-1}} \frac{1}{\|x-y\|^{n-3}} f(x) d\pi(x)$$

where π is the Lebesgue measure on S^{n-1} and $f(x) = \frac{1}{\|cx+d\|^{n+1}} h(C^{-1}(x))$ with c and d coefficents arising from the inverse Cayley transformation.

This would suggest that we are looking for a differential operator that annihilates the function $\frac{1}{\|x-y\|^{n-3}}$. As a first attempt we should ask if it is the case that $D_S^2 \frac{1}{\|x-y\|^{n-3}} = 0$. However a simple direct calculation reveals that

$$D_S \frac{1}{\|x-y\|^{n-3}} = G'(x-y) - \frac{x}{\|x-y\|^{n-3}}.$$

It follows that $(D_S + x)\frac{1}{\|x-y\|^{n-3}} = G'(x-y)$. Consequently

$$D_S(D_S + x)\frac{1}{\|x-y\|^{n-3}} = 0.$$

It follows that a suitable analogue of the euclidean Laplacian on the sphere is the differential operator $D_S(D_S + x)$. We shall denote this operator by Δ_S. The operator Δ_S is in fact a scalar valued differential operator. This follows by computing D_n^2 in terms of spherical co-ordinates. In this case the usual Laplace-Beltrami operator for the sphere factors as $((n-2)I - \Gamma)\Gamma$. Of course the Laplace-Beltrami operator is a scalar. So for each $\alpha \in \mathbf{C}$ the operator $-(\alpha - n + 2 + \Gamma)(\Gamma - \alpha)$ is a scalar. Moreover when $\alpha = \frac{n-1}{2}$ we get Δ_S.

Now $\Delta_S = D_S^2 + D_S x$. Furthermore $D_S = x(\Gamma - \frac{n-1}{2})x = x^2(-\Gamma + (n-1) - \frac{n-1}{2}) = -xD_S$. So Δ_S is also equal to $(D_S - x)D_S$. We may now use these two alternative forms for factoring Δ_S to obtain the following representation formula.

THEOREM 1. *[5] Suppose that U is a domain on S^{n-1} and $h : U \to Cl_n$ is a C^2 function. Suppose also that M is a piecewise smooth $(n-2)$-dimensional manifold lying in U and bounding a subdomain V of U. Then for each $y \in V$ we have*

$$h(y) = \frac{1}{\omega_{n-1}} \int_M (G'(x-y)n(x)h(x) - H'(x-y)n(x)D_S h(x))d\mu(x)$$
$$- \frac{1}{\omega_{n-1}} \int_V H'(x-y)\Delta_S h(x) d\pi(x),$$

where $H'(x-y) = \frac{1}{(n-3)\|x-y\|^{n-3}}$, $n(x)$ is the unit vector lying in the tangent space of S^{n-1} at x and orthogonal to the tangent space of M at x. Moreover $n(x)$ is outward pointing from M. Also μ is the Lebesgue measure on M.

Proof: Let $B_S(y, r)$ be a ball lying in S^{n-1}, centered at y and of radius r and let $S(y, r)$ denote its boundary. Applying Stokes' Theorem to the expression

$$\int_{M - S(y,r)} (G'(x-y)n(x)h(x) - H'(x-y)n(x)D_S h(x))d\mu(x)$$

gives the term

$$\int_{V\setminus B(y,r)}(G'(x-y)(D_S h(x))-(H'(x-y)D_S)D_S h(x)+H'(x-y)D_S^2 h(x))d\pi(x).$$

But

$$G'(x-y)(D_S h(x))-(H'(x-y)D_S)D_S h(x)+H'(x-y)(D_S^2 h(x)))$$

is equal to

$$G'(x-y)(D_S h(x)-(H'(x-y)(D_S+x))h(x)$$

$$+H'(x-y)xD_S h(x)-H'(x-y)(D_S-x)D_S h(x)-H'(x-y)xD_S h(x).$$

This expression reduces to $H'(x-y)(D_S-x)D_S h(x)$ or equivalently $H'(x-y)(D_S(D_S+x)h(x)$. The result now follows by allowing r to tend to zero. □

If we further assume that the function h is real valued then the non-scalar parts of our representation formula must be zero. The representation formula then reduces to

$$h(y)=\frac{1}{\omega_{n-1}}\int_M(<G'(x-y),n(x)>h(x)$$

$$-H'(x-y)<n(x),D_S h(x)>)d\mu(x)-\frac{1}{\omega_{n-1}}\int_V H'(x-y)\Delta_S h(x)d\pi(x)$$

where $<,>$ denotes the standard inner product on $\mathbf{R}^n$.

Now $<D_S,n(x)>=<x\Gamma-x\frac{n-1}{2},n(x)>$. As x and $n(x)$ are orthogonal then $<D_S,n(x)>=<x\Gamma,n(x)>$ and the previous representation formula reduces to

$$h(y)=\frac{1}{\omega_{n-1}}(\int_M(<G'(x-y),n(x)>h(x)$$

$$-H'(x-y)<x\Gamma h(x),n(x)>)d\mu(x)-\int_V H'(x-y)\Delta_S h(x)d\pi(x)).$$

Furthermore as $G'(x-y)=x\Gamma H'(x-y)-x(n-3)H'(x-y)$ the representation formula further reduces to

$$h(y)=\frac{1}{\omega_{n-1}}(\int_M(<x\Gamma H'(x-y),n(x)>h(x)$$

$$-H'(x-y)<x\Gamma h(x),n(x)>)d\mu(x)-\int_V H'(x-y)\Delta_S h(x)d\pi(x)).$$

If it is also the case that the scalar valued function h further satisfies the equation $\Delta h = 0$ then we obtain the following version of Green's formula on the sphere:

$$h(y) = \frac{1}{\omega_n} \int_M (< x\Gamma H'(x-y), n(x) > h(x)$$

$$-H'(x-y) < n(x), x\Gamma h(x) >) d\mu(x).$$

If $h : U \to Cl_n$ and $\Delta h = 0$ then we automatically get the following version of Green's formula.

$$h(y) = \frac{1}{\omega_{n-1}} \int_M (G'(x-y)n(x)h(x) - H'(x-y)n(x)D_S h(x) d\mu(x).$$

On the other hand if $h : U \to Cl_n$ is C^2 and $h|_M = 0$ then our representation formula becomes

$$h(y) = \frac{1}{\omega_{n-1}} \int_V H'(x-y)\Delta_S h(x) d\pi(x).$$

In this case we can use this representation formula to adapt arguments presented in the euclidean case in [?] and the Cayley transformation to show that Δ_S is conformally equivalent to Δ_{n-1} the Laplacian in $\mathbf{R}^n$. See [5] for details. Specifically

$$J_{-2}(C,x)\Delta_S h(y) = \Delta_{n-1} J_2(C,x) h(C(x))$$

where $C(x) = y$, $J_{-2}(C,x) = \frac{1}{\|cx+d\|^{n-1}}$ and $J_2(C,x) = \frac{1}{\|cx+d\|^{n-3}}$.

4. Other Operators and Representation Formulas

We have seen in the previous section that $D_S H'(x-y) = G'(x-y) - xH'(x-y)$. From this it is a simple matter to deduce the following Cauchy Integral Formula.

THEOREM 2. *Suppose that for U a domain on S^{n-1} the C^1 function $f : U \to Cl_n$ satisfies the equation $(D+x)f(x) = 0$. Suppose also that M and V are as in Theorem 1 and that $y \in V$. Then*

$$f(y) = \frac{1}{\omega_{n-1}} \int_M (G'(x-y) - xH'(x-y))n(x)f(x)d\mu(x).$$

It is also easy to deduce that if f is just a C^1 function then

$$f(y) = \frac{1}{\omega_{n-1}} (\int_M (G'(x-y) - xH'(x-y))n(x)f(x)d\mu(x)$$

$$-\int_V (G'(x-y) - xH'(x-y))(D_S + x)f(x)d\pi(x)).$$

If $f|_M = 0$ then this formula becomes

$$f(y) = \frac{1}{\omega_{n-1}}\int_V (G'(x-y) - xH'(x-y))(D_S + x)f(x)d\pi(x). \qquad (1)$$

These formulas follow as $(D_S - x)(G'(x-y) - xH'(x-y)) = 0$.

As we have previously observed if D_n is the euclidean Dirac operator over $\mathbf{R}^n$ with respect to a variable u, then D_n conformally transforms to $aD_n\tilde{a}$ where D_n in this last expression is the Dirac operator with respect to the vector variable v where $u = av\tilde{a}$ and $a \in Pin(n)$. It follows on splitting up D_n into its spherical and radial parts that if $y\Gamma$ is the spherical part with respect to a variable $y \in S^{n-1}$ then this operator is conformally equivalent to the operator $ax\Gamma\tilde{a}$ with respect to the variable $x \in S^{n-1}$. Consequently the operator D_S is conformally equivalent to $aD_S\tilde{a}$ under the same change of variables and the differential operators $D_S \pm y$ are conformally equivalent to $a(D_S \pm x)\tilde{a}$. For that matter the differential operator $D_S + \alpha y$ is conformally equivalent to $a(D_S + \alpha x)\tilde{a}$ for each $\alpha \in \mathbf{C}$.

If now $u = ax\tilde{a} \in S^{n-1}$ and $v = ay\tilde{a} \in S^{n-1}$ and $(D_S + v)f(v) = 0$ then on changing variables from u and v to x and y we get that

$$f(ay\tilde{a}) = \frac{1}{\omega_{n-1}}\int_{a^{-1}M\tilde{a}^{-1}} a(G'(x-y)-xH'(x-y))\tilde{a}an(x)\tilde{a}f(ax\tilde{a})d\mu(x),$$

where $a^{-1}M\tilde{a}^{-1} = \{x \in S^{n-1} : ax\tilde{a} \in M\}$.

DEFINITION 2. *Suppose that f is as in Theorem 2 so $(D_S+x)f(x) = 0$ then we say that f is left x monogenic.*

We can state a similar definition for right x monogenic functions.

The previous calculation tells us that $f(v)$ is left x monogenic if and only if $\tilde{a}f(ax\tilde{a})$ is left x monogenic. We have also established that the Dirac type operators $D_S + \alpha x$ and $D_S + \alpha u$ are intertwined by a and $\tilde{a}$. Moreover the convolution operator $(G(x-y) - xH(x-y)) \star |_M$ is also intertwined by the elements a and $\tilde{a}$ of $Pin(n)$. In fact the convolution operator $(G'(x-y) - xH'(x-y)) \star |_V$ is also intertwined by a and $\tilde{a}$. This may be noted by making the appropriate change of variable in Equation 1. By expanding V out to encompass all of S^{n-1} it may be determined that the convolution operator $(G(x-y)-xH(x-y))\star|_{S^{n-1}}$ is also intertwined by the a and $\tilde{a}$.

We already know from [8] that

$$J_{-1}(C,x)D_S = D_{n-1}J(C,x) \qquad (2)$$

where $J_{-1}(C,x) = \frac{\widetilde{cx+d}}{\|cx+d\|^{n+1}}$. Using this fact we want to determine what differential operator over R^{n-1} the differential operator $D_S + x$ is conformally equivalent to. First let us observe that the vector $x \in S^{n-1}$ is also the unit outer vector to S^{n-1} at x. Under the Cayley transformation this vector is transformed to $\frac{(cx+d)e_n\widetilde{(cx+d)}}{\|cx+d\|^2}$. It follows from Equation 2 that $(D_S + y)f(y)$ is transformed to

$$J_{-1}(C,x)^{-1}D_{n-1} + \frac{(cx+d)e_n\widetilde{(cx+d)}}{\|cx+d\|^2}f(C(x))$$

and this term simplifies to

$$J_{-1}(C,x)^{-1}(D_{n-1} + \frac{e_n}{\|cx+d\|^2})J(C,x)f(C(x)).$$

So

$$J_{-1}(C,x)(D_S + x)f(y) = (D_{n-1} + \frac{e_n}{\|cx+d\|^2})J(C,x)f(C(x))$$

where $y = C(x)$.

In slightly greater generality one also readily has

$$J_{-1}(C,x)(D_S + \alpha x)f(y) = (D_{n-1} + \alpha\frac{e_n}{\|cx+d\|^2})J(C,x)f(C(x)).$$

for any $\alpha \in \mathbf{C}$. This last equation bears a striking resemblance to a formula given in [10] describing the conformal covariance of the differential operator $D_{n-1} + m$ where $m \in \mathbf{R}^+$. In this case the operator satisfies the equation

$$J_{-1}(\psi,x)(D_{n-1} + m)f(y) = (D_{n-1} + \frac{m}{\|cx+d\|^2})J(\psi,x)f(\psi(x))$$

where here $y = \psi(x)$ and ψ is a Möbius transformation over $\mathbf{R}^{n-1} \cup \{\infty\}$. We would obtain the operator $D_{n-1} + \frac{1}{\|cx+d\|^2}$ instead of the operator $D_{n-1} + \frac{e_n}{\|cx+d\|^2}$ if we assumed that the unit 1 of Cl_n is the unit normal vector to $\mathbf{R}^{n-1}$ instead of e_n.

Returning to the Cauchy integral formula given in Theorem 2 it would now be natural to ask how this formula transforms under the Cayley transformation. Again we shall interpret the vector $x \in S^{n-1}$ as the unit outer vector to S^{n-1} at x. In this case the integral becomes

$$\frac{1}{\omega_{n-1}}\int_{C^{-1}(M)}(J(C,v)^{-1}G(u-v)J(C,u)^{-1}$$

$$-\frac{(u+e_n)e_n(u+e_n)}{\|u+e_n\|^2}\|v+e_n\|^{n-3}H(u-v)\|u+e_n\|^{n-3})$$

$$J(C,u)n(u)J(C,u)f(C(u))d\sigma(u).$$

This expression simplifies to

$$J(C,v)^{-1}\frac{1}{\omega_{n-1}}\int_{C^{-1}(M)}(G(u-v)$$

$$-(v+e_n)(u+e_n)^{-1}e_n)n(u)J(C,u)f(C(u))d\sigma(u).$$

As $ue_n = -e_n u$ for each $u \in \mathbf{R}^{n-1}$ the term $(v+e_n)(u+e_n)^{-1}e_n$ is equal to $(v+e_n)e_n(e_n-u)^{-1}$. If we place $G((u-v)+(v+e_n)e_n(u-e_n)^{-1}H(u-v) = Q(u,v)$ then we have established the following:

THEOREM 3. *Suppose U is a domain on S^{n-1} then a C^1 function $f : U \to Cl_n$ satisfies the equation $(D_S + x)f(x) = 0$ on U if and only if*

$$(D_{n-1} + \frac{e_n}{\|cx+d\|^2})J(C,u)f(C(u)) = 0$$

on $C^{-1}(U) \subset \mathbf{R}^{n-1}$. Moreover, if $v \in C^{-1}(V)$ and V is as in Theorem 1 then

$$J(C,v)f(C(v)) = \frac{1}{\omega_{n-1}}\int_{C^{-1}(M)} Q(u,v)n(u)J(C,u)f(C(u))d\sigma(u).$$

Let us return momentarily to the kernel $G'(x-y)$ and the operator D_S. One may use the Cauchy integral formula given in [8, 9] together with standard techniques described in [6] and elsewhere to obtain the following:

THEOREM 4. *Suppose that Σ is the boundary of a strongly Lipschitz domain in $\mathbf{R}^{n-1}$ and $\Pi = C(\Sigma)$ then for $1 < p < \infty$*

$$L^p(\Pi) = H^p(\Pi^+) \oplus H^p(\Pi^-)$$

where $L^P(\Pi)$ is the space of Cl_n valued L^p integrable functions defined on Π, $\Pi^\pm$ are the domains in S^{n-1} which complement Π and $H^p(\Pi^\pm)$ are the Hardy spaces of solutions to the equation $D_S f(x) = 0$ on $\Pi^\pm$ that have L^p non-tangential limit functions defined on Π.

Suppose now that f belongs to the Hardy space $H^p(\Pi^+)$ then $D_S(D_S - x)f(x) = 0$. so each component of f is annihilated by $\triangle_S$. In particular the identity component of f is annihilated by this operator.

Let us now consider the special case where Π is the equator S^{n-2} of S^{n-1}. So $S^{n-2} = S^{n-1} \cap \mathbf{R}^{n-1}$. Let us also assume that Π^+ is the hemisphere $S^{n-2,+}$ that has boundary S^{n-2} and contains the point e_n. For $\lambda(x) \in L^p(S^{n-2})$, with $1 < p < \infty$, the function

$$\frac{1}{\omega_{n-1}}\int_{S^{n-2}} G'(x-y)n(x)\lambda(x)d\mu(x) \qquad (3)$$

defines a member of $H^p(S^{n-2,+})$. Moreover as y tends non-tangentially to a point $z \in S^{n-2}$ this integral approaches the value

$$\frac{1}{2}\lambda(z) + P.V.\frac{1}{\omega_{n-1}}\int_{S^{n-2}} G'(x-z)n(x)\lambda(x)d\mu(x)$$

almost everywhere and the function defined by this integral belongs to $L^p(S^{n-2})$.

As in this case $n(x) = e_n$ we get when λ is scalar valued then the scalar part of the integral 3 is given by

$$\frac{1}{\omega_{n-1}}\int_{S^{n-2}} P(x-y)\lambda(x)d\mu(x)$$

where $P(x,y)$ is the n-th component $\frac{y_n}{\|x-y\|^{n-1}}$ of $G'(x-y)$. So on multiplying this integral by 2 gives a solution to the Dirichlet problem on $S^{n-2,+}$ for $1 < p < \infty$ and the kernel $2P$ is the analogue of the Poisson kernel in upper half space, but now in the context of the hemisphere.

In conclusion let us place $G(x-y) + xH(x-y) = Q'(x,y)$. Then if h is a C^2 function on U and $\triangle_S h = 0$ and V, M and y are as in Theorem 1 one may apply Stokes' Theorem to obtain the following representation formula.

$$h(y) = \frac{1}{\omega_{n-1}}\int_M (Q'(x,y)n(x)h(x) - H'(x-y)n(x)(D+x)h(x))d\mu(x).$$

This formula gives a second and alternative representation for solutions to the equation $\triangle_S h = 0$. However if h is scalar valued then we only real need the scalar part of this integral formula and one may readily determine that this reduces to the scalar version of Green's formula that we presented earlier in this paper.

In [8, 9] we used an alternative Cayley transformation, to set up analogous results on the hyperbola to those obtained on the sphere. It is a simple exercise to see that all the results presented here and in [5] also carry over to the context of the hyperbola.

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