The Quaternionic Riemann problem

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ABSTRACT. We consider a generalization of the classical Riemann problem in complex analysis. We outline some similarities between both problems and main differences. Especially, we prove that under simple conditions the index of the problem is zero and uniquely solvable in L_2. We show a connection between the index and the wrapping number.

1. Introduction

The classical Riemann problem in complex analysis, often also called Hilbert or Riemann-Hilbert problem, has a lot of relations to several topics in Mathematics and its applications.

1.1. The complex Riemann problem. It consists in dermining a function f which is holomorphic in $\mathbb{C}\backslash\Gamma$, where $\Gamma = \partial D_-$ and D_- is a bounded and simply connected domain in the complex plane and D_+ is its unbounded open complement. Further f satiesfies the boundary condition

$$a(z)f_-(z) = b(z)f_+(z) + h(z) \text{ on } \Gamma \tag{1}$$

and $f(z) \to \infty$ as $z \to \infty$. Here, $a(z), b(z), h(z)$ are given functions on Γ, $f_-(z)$ and f_+ are the boundary values of $f(z)$ from D_- or D_+ respectivly.

Using the singular Cauchy integral operator

$$(Su)(z) = \frac{1}{i\pi}\int_\Gamma \frac{u(\zeta)}{z-\zeta}d\zeta$$

and associated projections

$$P = \tfrac{1}{2}(I+S) \text{ and } Q = \tfrac{1}{2}(I-S)$$

and Plemelj-Sokhotzki formulae we are able to refomulate (1) as

$$a(z)(Pf)(z) + b(z)(Qf)(z) = h(z) \tag{2}$$

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or only in term of the Cauchy operator

(3) $$\frac{(a(z)+b(z))}{2}f(z)+\frac{(a(z)-b(z))}{2}(Sf)(z)=h(z)$$

Equation (2) gave rise to an extended consideration of so-called paired operators which are studied by S. Prößdorf ([**Pr**], [**MP**]).

Equation (3) emphasizes the singular integral equation with Cauchy operator. Linear and also nonlinear equations of this type on the real line or equivalently on the unit circle are well studied for example by L. von Wolfersdorf ([**vW**]).

Singular integral operator on Lipschitz surfaces are investigated using Clifford analytical methods by by McIntosh and co-workers in [**MLS**] and [**MLQ**].

1.2. The quaternionic Riemann problem. Quaternions und quaternionic analysis are a higher dimensional generalization of complex numbers and complex analysis. The important $\bar{\partial}$-operator of complex analysis is replaced by the so-called Dirac operator

$$D=\sum_{j=1}^{3}e_j\frac{\partial}{\partial x_j},$$

where e_j are the generating vectors of quaternions. Quaternionic valued functions u which fulfill the equation $Du=0$ are called *monogenic* functions. The Dirac operator is a first order linear differential operator and thus has a fundamental solution. Such a fundamental solution can be used to create a singular Cauchy integral operator in quaternionic analysis. It turns out that the Dirac operator, its fundamental solution, the Cauchy operator and other related operators play the same role like the classical operators in complex analysis. Therefore it seems to be natural to state a Riemann problem in the quaternionic context. We want to restrict ourselfs to the following case:

Determine a function u which is monogenic in the upper half space $\mathbb{R}^3_+$ and fulfills the boundary condition

(4) $$a(x)u_-(x)=b(x)u_+(x)+h(x)\text{ on }\mathbb{R}^2,$$

and $|u(x)|=\mathcal{O}(|x|^{-1/2})$ as $|x|\to\infty$. Here $a(x),b(x),h(x)$ are quaternionic valued function on $\mathbb{R}^2$.

Because the only conformal transformations in higher dimensions are Möbius transformation it is unclear up to now if this situation can be translated in a more or less simple way to a larger class of manifolds.

Nevertheless equation (4) can be reformulated as an singular integral equation like equation (1). Based on considerations about singular integral operators the Riemann problem in a quaternionic setting was first studied by M.V. Shapiro and N.L. Vasilewski ([**Sh1**], [**Sh2**], [**ShV**]) which considered algebras of singular integral operators and Riemann boundary value problems and obtained conditions for Fredholmean operators. A special result states when the Riemann problem is Fredholmean.

A special case of the Riemann problem in L_2 was considered in [**Be1**] where a simple condition for solving via successive approximation is given. Some nonlinear singular integral equations involving the Cauchy integral operator are treated in [**Be2**] and [**Be3**].

In this paper we will show that under simple assumptions the index of the Riemann problem is zero and that the problem is, under some conditions, uniquely solvable in $L_2(\mathbb{R}^2)$. Moreover, we want to mention a connection between the index of the Riemann problem and a topological invariant - the wrapping number. This wrapping number also called winding number was considered by A. Sudbery in [**Su**], by K. Habetha in [**Ha**] and in a larger context by D. Hestenes and G. Sobzcyk in [**HS**].

2. Preliminaries

We shall briefly review some basic definitions and properties of function theory corresponding to the quaternions. Meanwhile there exist a lot of good books concerning Clifford and quaternionic analysis. The basic book for Clifford analysis is [**BDS**], a comprehensive outline is also contained in [**DSS**] which deals with the function theory of the Dirac operator and related special functions and also with residual theory. A more harmonic analytical viewpoint is given in [**GM**]. A summary of developments in Clifford and quaternionic analysis and its relations to physics and some numerical stuff is contained in [**GS2**]. The books [**GS1**] and [**KS**] deal with quaternionic analysis. Both outline physical problems, the second in more detailed then the first one.

2.1. Algebra of quaternions $\mathbb{H}$. We denote be e_1, e_2, e_3 the generating elements of the quaternions and be e_0 the unit element. Then these elements fullfills the relations

$$e_i e_j + e_j e_i = -2\delta_{ij} e_0, \ i, j \neq 0,$$
$$e_1 e_2 = e_3, \ e_2 e_3 = e_1, \ e_3 e_1 = e_2.$$

An arbitrary element $q \in \mathbb{H}$ is given by

$$q = q_0 e_0 + \sum_{j=1}^{3} q_j e_j = \mathcal{S}c\, q + \mathcal{V}ec\, q$$

and the conjugated quaternion by

$$\bar{q} = q_0 e_0 - \sum_{j=1}^{3} q_j e_j = \mathcal{S}c\, q - \mathcal{V}ec\, q.$$

We have $q\bar{q} = \sum_{k=0}^{3} q_k^2 = |q|^2$. If the coefficients $q_k \in \mathbb{R}$ then q is an element of the algebra of real quaternions $\mathbb{H}(\mathbb{R})$ and $(q\bar{q})^{1/2} = |q|$ is the length of the vector $\vec{q} \in \mathbb{R}^4$. If $q_k \in \mathbb{C}$ then q is an element of the algebra of complex quaternions $\mathbb{H}(\mathbb{C})$. The complex unit i commutes with all elements e_j. In the complex quaternions we can define another conjugation

$$\tilde{q} = \bar{q}_0 e_0 - \sum_{j=1}^{3} \bar{q}_j e_j,$$

where $\bar{q}_k$ denotes the complex conjugated to q_k.

An elment $q \in \mathbb{H}(\mathbb{R})$ is invertible if and only if

$$q\bar{q} \neq 0 \tag{5}$$

and the inverse element is given by

$$q^{(-1)} = \frac{\bar{q}}{q\bar{q}}. \tag{6}$$

Furthermore, all elements $q \in \mathbb{H}(\mathbb{R})$, $q \neq 0$, are invertible. If $q \in \mathbb{H}(\mathbb{C})$ condition (5) is still necessary and sufficient for q to be invertible and the inverse element in this case is also given by (6). This can be seen by the

Matrix representation. quaternions from $\mathbb{H}(\mathbb{R})$ and $\mathbb{H}(\mathbb{C})$ may be represented as real- or complex 4×4-matrices:

$$\sum_{k=0}^{3} q_k e_k = q \sim \begin{pmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{pmatrix} = Q$$

A simple computation shows that

$$|\det Q| = |q\bar{q}|^2. \tag{7}$$

But in $\mathbb{H}(\mathbb{C})$ there exist zero divisors!

EXAMPLE. Let $q = 1 + ie_3$ then $\bar{q} = 1 - ie_3$ and

$$q\bar{q} = (1 + ie_3)(1 - ie_3) = 1 + ie_3 - ie_3 - i^2 e_3^2 = 1 - 1 = 0.$$

Thus $1 + ie_3$ and also $1 - ie_3$ are not invertible.

2.2. Quaternionic analysis. Let $G \subset \mathbb{R}^3$ be a domain with smooth boundary Γ. We identify $(x_1, x_2, x_3) = \vec{x} \in \mathbb{R}^3$ with $x = \sum_{j=1}^{3} x_j e_j \in \mathbb{H}$. We consider functions f defined in G with values in $\mathbb{H}(\mathbb{C})$. These functions may be written as

$$f(x) = \sum_{k=0}^{3} f_k(x) e_k, \; f_k(x) \in \mathbb{C}, \; x \in G.$$

Properties such as continuity, differentiability, and so on, which are ascribed to f have to be possesed by all components $f_k(x), k = 0, 1, 2, 3$. In this way the usual Banach spaces of these functions are denoted by C^α, L_p and W_p^k. In $L_2(G)$ there exists an $\mathbb{H}$-valued inner product

$$(u, v) = \int_G \bar{u}(x) v(x) \, dG_x.$$

As mentioned before the Dirac operator is defined by

$$D = \sum_{j=1}^{3} e_j \frac{\partial}{\partial x_j}.$$

A function f which fulfills $Df = 0$ in G is called *(left)-monogenic* in G. Because the multiplication in the algebra of quaternions isn't commutative we have to deside between left- and right-monogenic. We only want to use left-monogenicity and write for this situation only "monogenic". We define the Cauchy-kernel in $\mathbb{R}^3$ by

$$e(x) = \frac{-x}{4\pi |x|^3}, \; x \neq 0.$$

$e(x)$ is a fundamental solution of D and therefore monogenic in $\mathbb{R}^3\backslash\{0\}$. Using $e(x)$ we introduce the following integral operators

$$(T_G u)(x) := -\int_G e(x-y)u(y)\,dy, \qquad x \in \mathbb{R}^3 \qquad \text{(Teodorescu transform)}$$

$$(F_\Gamma u)(x) := \int_\Gamma e(x-y)n(y)u(y)\,d\Gamma_y, \quad x \notin \Gamma \qquad \text{(Cauchy-type operator)}$$

$$(S_\Gamma u)(x) := \int_\Gamma 2e(x-y)n(y)u(y)\,d\Gamma_y, \quad x \in \Gamma \qquad \text{(singular Cauchy operator)}$$

where $n(y) = \sum_{j=1}^{3} n_j(y)e_j$ is the outward pointing unit vector to Γ at the point y. From ([**GS1**]) we get immediatly the following statements:

LEMMA 1. *Let $u \in C^1(G) \cap C(\bar{G})$. Then*

$$(F_\Gamma u)(x) + (T_G D)u(x) = \begin{cases} u(x), & x \in G, \\ 0, & x \in \mathbb{R}^3\backslash\bar{G} \end{cases} \quad \textit{(Borel-Pompeiu formula)}$$

$$(DT_G u)(x) = \begin{cases} u(x), & x \in G, \\ 0, & x \in \mathbb{R}^3\backslash\bar{G} \end{cases} \quad (T_G \textit{ is right-invers to } D)$$

$$(DF_\Gamma)u(x) = 0, \ x \in \mathbb{R}^3\backslash\Gamma \quad (F_\Gamma \textit{ represents monogenic functions}).$$

LEMMA 2 (Plemelj-Sokhotzkij's formulae). *Let $u \in C^{0,\alpha}(G), 0 < \alpha < 1$. Then*

$$(i) \lim_{\substack{x\to\xi\in\Gamma \\ x\in G}} (F_\Gamma u)(x) = P_\Gamma u(\xi), \quad (ii) \lim_{\substack{x\to\xi\in\Gamma \\ x\in\mathbb{R}^3\backslash\bar{G}}} (F_\Gamma u)(x) = -Q_\Gamma u(\xi)$$

for any $\xi \in \Gamma$.

Here, the operator $P_\Gamma := \frac{1}{2}(I+S)$ denotes the projection onto the space of all $\mathbb{H}$-valued functions which have a left-monogenic extension into the domain G, $Q_\Gamma := \frac{1}{2}(I-S)$ denotes the projection onto the space of all $\mathbb{H}$-valued functions which have a left-monogenic extension into the domain $\mathbb{R}^3\backslash\bar{G}$ and vanish at infinity. We have the direct decomposition $(1 < p < \infty)$

$$L_p(\Gamma) = \operatorname{im} P_\Gamma \cap L_p(\Gamma) \oplus \operatorname{im} Q_\Gamma \cap L_p(\Gamma)$$

and the relations

$$S_\Gamma P_\Gamma = P_\Gamma, \ S_\Gamma Q_\Gamma = -Q_\Gamma, \ S_\Gamma^2 = S_\Gamma S_\Gamma = I.$$

From complex analysis it is well-known that the product of two analytic functions is again an analytic function and if a function is a boundary value of a monogenic function then the product of this function with a constant is also a boundary value of an anlytic function. We will see that this is i.g. not true the quaternionic case.

EXAMPLE. The product of two monogenic functions need not to be a monogenic function. Let be $u = x_1e_2 + x_2e_1$ and $v = x_1e_1 - x_2e_2$ then $Du = e_1e_2 + e_2e_1 = 0$

and $Dv = e_1^2 - e_2^2 = 0$. But

$$uv = (x_1e_2 + x_2e_1)(x_1e_1 - x_2e_2) = x_1^2e_2e_1 + x_2x_1e_1^2 - x_1x_2e_2^2 - x_2^2e_1e_2 =$$
$$= -x_1^2e_1e_2 - x_1x_2 + x_1x_2 - x_2^2e_1e_2 = -(x_1^2 + x_2^2)e_3$$

and $D(uv) = -2x_1e_1e_3 - 2x_2e_2e_3 = 2x_1e_2 - 2x_2e_1$.

EXAMPLE. If u fulfills $S_{\mathbb{R}^2}u = u$ on $\mathbb{R}^2$ which means $u \in \operatorname{im} P_{\mathbb{R}^2}$ then $S_{\mathbb{R}^2}(e_3u) = -e_3u$ on $\mathbb{R}^2$ and thus $e_3u \in \operatorname{im} Q_{\mathbb{R}^2}$. This is easily seen from

$$(S_{\mathbb{R}^2}e_3u)(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(y_1 - x_1)e_1 + (y_2 - x_2)e_2}{|x-y|^3} e_3e_3u(y)dy =$$
$$= -e_3\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(y_1 - x_1)e_1 + (y_2 - x_2)e_2}{|x-y|^3} e_3u(y)dy = -e_3u(x).$$

3. Singular Integral equations

In what follows we will briefly write S, P, Q instead of $S_{\mathbb{R}^2}$ and so on. Using Plemelj-Sokhotzkij's formulae we can reformulate the quaternionic Riemann problem (4) in terms of the Hardy projections, i.e.

$$a(x)(Pu)(x) + b(x)(Qu)(x) = h(x) \text{ on } \mathbb{R}^2, \tag{8}$$

or in terms of the singular Cauchy intergal operator

$$\tfrac{(a(x)+b(x))}{2}u(x) + \tfrac{(a(x)-b(x))}{2}(Su)(x) = h(x) \tag{9}$$

we simplify this to

$$c(x)u(x) + d(x)(Su)(x) = h(x) \text{ on } \mathbb{R}^2. \tag{10}$$

3.1. Fredholmness, Index and Symbol.

Fourier transform is defined as

$$\hat{w}(\xi) = \mathcal{F}w := \lim_{\epsilon\to 0,\ N\to\infty} \int_{\epsilon<|x|<N} e^{-i(x,\xi)}w(x)dx.$$

For the convolution

$$v(x) = \int_{\mathbb{R}^n} K(x-y)u(y)dy$$

the definition leads to the formula

$$\hat{v}(\xi) = \hat{K}(\xi)\hat{u}(\xi)$$

For the singular integral operator

$$(A_0u)(x) = v(x) := au(x) + \int_{\mathbb{R}^n} K(x-y)u(y)dy,$$
$$K(x-y) = |x-y|^{-n} f(\theta), \theta = (y-x)|x-y|^{-1} \in S^{n-1},$$

which can be considered as a convolution of $a\delta + K$ and the function u, we may define the symbol of A_0 by

$$\Phi(\xi) := SymbA_0 = a + \hat{K}(\xi)$$

and we have

$$v = A_0 u = \mathcal{F}^{-1}\Phi(\xi)\mathcal{F}u.$$

To get a formula for singular integral operators with a characteristic depending on the pole, i.e. $f = f(x,\theta)$, we consider the singular integral operator

$$(A_t u)(x) := a(t)u(x) + \int_{\mathbb{R}^n} K(t, x-y)u(y)dy,$$

that depends on a parameter $t \in \mathbb{R}^n$ and the symbol of which is given by

$$SymbA_t = \Phi(t,\xi) = a(t) + \hat{K}(t,\xi)$$

and

$$A_t = \mathcal{F}\Phi(t,\xi)\mathcal{F}.$$

Now, putting $t = x$ we obtain that the singular integral operator

$$(Au)(x) := a(x)u(x) + \int_{\mathbb{R}^n} K(x, x-y)u(y)dy,$$
$$K(x, x-y) = f(x,\theta)|x-y|^{-n}, \ \theta \in S^{n-1}$$

has the symbol $\Phi(x,\xi)$ and may be written as

$$A = \mathcal{F}^{-1}\Phi(x,\xi)\mathcal{F}.$$

The symbol can be used to determine if a singular integral operator is a Fredholm operator. Moreover, it will be used to explain when the index is zero.

DEFINITION. If at least one of the dimensions

$$\alpha(A) = \dim\ker A \quad \text{and} \quad \beta(A) = \dim\operatorname{coker} A$$

is finite, their difference is called the *index* of A and denoted by

$$\operatorname{ind} A = \alpha(A) - \beta(A).$$

A normally solvable operator A is called a *Fredholm operator* if it has finite index. Obviously, $\operatorname{ind} A$ is finite if and only if both $\alpha(A)$ and $\beta(A)$ are finite.

LEMMA 3. *Let $\mathcal{R}$ be the restriction of the ring of symbols $\Phi(x,\theta)$ of singular integral operators such that*

1. $\Phi(x,\theta) \in W_2^l(S^1)$ *uniformly with respect to x and*

$$l > \begin{cases} \frac{1}{2}, & p = 2, \\ \frac{1}{q} + \frac{1}{2}, & p \neq 2, q = \min(p,p'), \ \frac{1}{p} + \frac{1}{p'} = 1 \end{cases}$$

2. *The symbol $\Phi(x,\theta)$ is continuous on $\overline{\mathbb{R}^2}$ uniformly with respect to θ;*

which are continuous on $\overline{\mathbb{R}^2} \times S^1$.
Then a singular operator A with symbol $\Phi(x,\theta) \in \mathcal{R}$ is a Fredholm operator in $L_p(\mathbb{R}^2), 1 < p < \infty$, if and only if

$$\inf_{(x,\theta)} |\Phi(x,\theta)\overline{\Phi(x,\theta)}| > 0. \tag{11}$$

PROOF. The proof can be obtained from **[MP]**, especially from Theorem 8.3., page 271. We only remember that for quaternions fulfill the relation (7). This Lemma may also be proved using ideas developed in **[ShV]**. □

3.2. Equations with coefficients in $\mathbb{H}(\mathbb{R})$.

THEOREM 1. *Let $c(x)$, $d(x) \in C(\overline{\mathbb{R}^2}, \mathbb{H}(\mathbb{R}))$ then*

$$c(x)I + d(x)S$$

is a Fredholm operator in $L_p(\overline{\mathbb{R}^2}, \mathbb{H}(\mathbb{R}))$ and has index zero if and only if

$$\inf_x ||c(x)|^2 - |d(x)|^2| > 0.$$

PROOF OF FREDHOLMNESS. This result was obtained in **[ShV]**.

The singular integral operator $c(x)I + d(x)S$ has the symbol

$$\Phi(x,\theta) = c(x) + id(x)\theta e_3, \text{ where } \theta = \theta_1 e_1 + \theta_2 e_2 \in S^1.$$

Thus

$$\begin{aligned} \Phi(x,\theta)\overline{\Phi(x,\theta)} &= |c(x)|^2 - |d(x)|^2 + i(c(x)\overline{d(x)\theta e_3} + \overline{c(x)}d(x)\theta e_3) = \\ &= |c(x)|^2 - |d(x)|^2 + 2i\langle B(x), \theta\rangle \end{aligned}$$

For any fixed x there exists an θ^* such that $\langle B(x), \theta^*\rangle = 0$. Therefore

$$\inf_{x,\theta} |\Phi(x,\theta)\overline{\Phi(x,\theta)}| = \inf_x ||c(x)|^2 - |d(x)|^2|$$

and this operator is a Fredholm operator if and only if

$$\inf_x ||c(x)|^2 - |d(x)|^2| > 0.$$

□

We want to interprete this in terms of the Hardy projections P and Q. This means that the operator

$$a(x)Pu + b(x)Q, \ a(x), b(x) \in C(\overline{\mathbb{R}^2}, \mathbb{H}(\mathbb{R}))$$

is a Fredholm operator if and only if

$$\inf_x \left||a(x)+b(x)|^2 - |a(x)-b(x)|^2\right| > 0.$$

We have

$$\begin{aligned}\left||a+b|^2 - |a-b|^2\right| &= |(a+b)(\bar{a}+\bar{b}) - (a-b)(\bar{a}-\bar{b})| = \\ &= 2|a\bar{b}+b\bar{a}| = 2|\langle a,b\rangle| = 2|a|\,|b|\,|\cos(a,b)|.\end{aligned}$$

Thus, the operator $a(x)Pu + b(x)Q$ is a Fredholm operator if and only if

$$a(x)\neq 0,\ b(x)\neq 0 \text{ and } \cos(a(x),b(x))\neq 0,\ \forall x.$$

EXAMPLE.

$$\text{The equation} \qquad Pu + Qu = h$$

is uniquely solvable with $u = h$. This follows from the direct decomposition

$$L_p(\mathbb{R}^2) = \operatorname{im} P \cap L_p(\mathbb{R}^2) \oplus \operatorname{im} Q \cap L_p(\mathbb{R}^2)$$

and we have

$$a(x) = b(x) = 1 \neq 0 \text{ and } \cos(1,1) = 1 \neq 0.$$

EXAMPLE. The equation $Pu + e_3Qu = h$ is equivalent to $Pu + P(e_3u) = h$, because of $e_3Su = -S(e_3u)$ and we obtain the equation $P(1+e_3)u = h$ which is solvable if and only if $h \in \operatorname{im} P$.
Thus the operator is not a Fredholm operator which can also be seen by $\cos(1,e_3) = 0$.

LEMMA 4 ([**MP**]). *Let the singular operator A_t with symbol $\Phi(x,\theta,t)$ depend on a parameter $t \in [0,1]$. Assume that*

1. $\Phi(x,\theta,t) \in W_2^l(S^1)$ *uniformly with respect to x and*
$$l > \begin{cases} \frac{1}{2}, & p = 2,\\ \frac{1}{q}+\frac{1}{2}, & p\neq 2, q = \min(p,p'),\ \frac{1}{p}+\frac{1}{p'} = 1\end{cases}$$
2. $||\Phi(x,\theta,t) - \Phi(x,\theta,t')||_{W_2^l} \to 0,$ *as $t' \to 0$, uniformly with respect to x and t;*
3. *The symbol $\Phi(x,\theta,t)$ is continuous on $\overline{\mathbb{R}^2}\times[0,1]$ uniformly with respect to θ;*
4. $\Phi(x,\theta,t)\overline{\Phi(x,\theta,t)} \in \mathbb{C}\backslash\{0\}$ *and* $\inf|\Phi(x,\theta,t)\overline{\Phi(x,\theta,t)}| > 0.$

Then $\operatorname{ind} A_t$ *does not depend on t.*

We will use this Lemma to prove that under the assumptions of Theorem 1 the index is zero.

PROOF OF THE INDEX PART. The condition $\inf_x \left||c(x)|^2 - |d(x)|^2\right| > 0$ and the continuity of the functions $c(x)$ and $d(x)$ lead to two cases:

The first case is $|c(x)|^2 - |d(x)|^2 > 0\ \forall x$, which implies $|c(x)| > 0\ \forall x$, here, we define a homotopy by

$$\Phi(x,\theta,t) = \sqrt{|c(x)|^2 - |d(x)|^2 + (1-t)^2|d(x)|^2}\,\frac{c(x)}{|c(x)|} + i(1-t)d(x)\theta e_3.$$

where

$$\overline{\Phi(x,\theta,t)}\Phi(x,\theta,t) = \left(|c(x)|^2 - |d(x)|^2 + (1-t)^2|d(x)|^2\right)\frac{|c(x)|^2}{|c(x)|^2} - (1-t)^2|d(x)|^2 + i\ldots$$
$$= |c(x)|^2 - |d(x)|^2 + i\ldots$$

thus $\Phi(x,\theta,t)$ is a Fredholm operator for all t, $0 \le t \le 1$.

Hence $\Phi(x,\theta) = \Phi(x,\theta,0)$ is homotopic to

$$\Phi(x,\theta,1) = \sqrt{|c(x)|^2 - |d(x)|^2}\frac{c(x)}{|c(x)|}$$

The other case is $|c(x)|^2 - |d(x)|^2 < 0\ \forall x$, which implies $|d(x)| > 0\ \forall x$, and we define a homotopy in two steps by

$$\Phi_1(x,\theta,t) = i\sqrt{|d(x)|^2 - |c(x)|^2 + (1-t)^2|c(x)|^2}\frac{d(x)}{|d(x)|}\theta e_3 + (1-t)c(x)$$
$$\text{and } \Phi_2(x,\theta,t) = i\sqrt{|d(x)|^2 - |c(x)|^2}\frac{d(x)}{|d(x)|}\cdot$$
$$\cdot\left(\cos\left(\frac{\pi}{2}(1-t)\right) + \theta e_3 \sin\left(\frac{\pi}{2}(1-t)\right)\right).$$

Here, we see that $\Phi(x,\theta) = \Phi_1(x,\theta,0)$ is homotopic to

$$\Phi_2(x,\theta,1) = i\sqrt{|d(x)|^2 - |c(x)|^2}\frac{d(x)}{|d(x)|}.$$

In both cases the symbol is homotopic to the symbol of an operator of multiplication with a non-vanishing continuous function. Thus the index must be zero. □

COROLLARY. *Let* $a(x), b(x) \in C(\overline{\mathbb{R}^2}, \mathbb{H}(\mathbb{R}))$ *then* $a(x)Pu + b(x)Q$ *is a Fredholm operator in* $L_p(\overline{\mathbb{R}^2}, \mathbb{H}(\mathbb{R}))$ *and has index zero if and only if* $a(x) \neq 0$, $b(x) \neq 0$ *and* $\cos(a(x), b(x)) \neq 0$, $\forall x$.

In case of $p = 2$ we can improve our result.

THEOREM 2. *Let* $c(x)$, $d(x) \in C(\overline{\mathbb{R}^2}, \mathbb{H}(\mathbb{R}))$ *then*

$$c(x)u(x) + d(x)(Su)(x) = h(x) \text{ on } \mathbb{R}^2$$

is uniquely solvable in $L_2(\overline{\mathbb{R}^2}, \mathbb{H}(\mathbb{R}))$ *for arbitrary* $h \in L_2(\overline{\mathbb{R}^2}, \mathbb{H}(\mathbb{R}))$ *iff*

$$\inf_x \left||c(x)|^2 - |d(x)|^2\right| > 0. \tag{12}$$

PROOF. Because under these assumptions the index is zero we only need to prove the uniqueness of the solution. The space $L_2(\overline{\mathbb{R}^2}, \mathbb{H}(\mathbb{R}))$ is a (real) Hilbert space with the scalar product

$$(u,v) = \mathcal{S}c\int_{\mathbb{R}^2} \overline{u(x)}v(x)\,dx, \quad ||u||^2 = (u,u).$$

It is not difficult to prove that S is self-adjoint, i.e. $S^* = S$ and with $S^2 = I$ we get $||S|| = 1$. The condition (12) leads to two cases: First, we assume $|c(x)|^2 - |d(x)|^2 >$

0, $\forall x$, thus $|c(x)|^2 > |d(x)|^2 \geq 0$, and $c^{(-1)}(x)$ exists and is well-difined for all x. Now, the equation

$$c(x)u + d(x)Su = 0 \text{ is equivalent to } \ u = -c^{(-1)}(x)d(x)Su$$

From this we conclude that

$$u = \big(-c^{(-1)}(x)d(x)S\big)^m u, \ m \in \mathbb{N},$$

and thus

$$||u|| = ||\big(-c^{(-1)}(x)d(x)S\big)^m u|| \leq |c^{(-1)}(x)d(x)|^m \to 0 \text{ as } m \to \infty$$

which leads to

$$||u|| = 0 \text{ and } u = 0.$$

in the second case, we get $|d(x)|^2 > |c(x)|^2 \geq 0$, $\forall x$, and $d^{(-1)}(x)$ is well-defined for all x. Here,

$$c(x)u + d(x)Su = 0 \text{ is equivalent to } \ -d^{(-1)}c(x)u = Su$$

using $S^2 = I$ we get

$$u = -S\big(d^{(-1)}(x)c(x)u\big)$$

and analogously to the first case we obtain

$$||u|| = 0 \text{ and } u = 0.$$

□

COROLLARY. *Let* $a(x)$, $b(x) \in C(\overline{\mathbb{R}^2}, \mathbb{H}(\mathbb{R}))$ *then*

$$a(x)Pu + b(x)Qu = h(x) \ \text{on } \mathbb{R}^2$$

is uniquely solvable in $L_2(\overline{\mathbb{R}^2}, \mathbb{H}(\mathbb{R}))$ *for arbitrary* $h \in L_2(\overline{\mathbb{R}^2}, \mathbb{H}(\mathbb{R}))$ *iff* $a(x) \neq 0$, $b(x) \neq 0$ *and* $\cos(a(x), b(x)) \neq 0$, $\forall x$.

REMARK. The ideas of the proof of Theorem 2 can be used to prove this theorem directly be successive approximation.

3.3. Equations with coefficients in $\mathbb{H}(\mathbb{C})$. If the coefficients are elements of the complex algebra of quaternions we are not able to get such simple conditions as in the previous section. But we know from Lemma 3, when we have a Fredholm operator and we will derive a certain mapping which descibes the index of the problem. We will use

The wrapping number. It generalizes the winding number in the complex plane to higher dimensions. Therefore it is also often called winding number.

DEFINITION. Let $\mathcal{N}$ be a closed hypersurface in $\mathbb{R}^4$ then the wrapping number (winding number) of $\mathcal{N}$ arround the point $y \notin \mathcal{N}$ is defined by

$$\#_y(\mathcal{N}) = \frac{1}{2\pi^2} \int_{\mathcal{N}} \frac{(x-y)}{|x-y|^4} n(x) d\mathcal{N}.$$

If f is a one-to-one mapping of a closed hypersurface $\mathcal{N}'$ onto $\mathcal{N}$ in $\mathbb{R}^4$, we can define a winding number for f by identifying with the wrapping number (winding number) of $\mathcal{N}$. We have (see **[HS]**)

$$\#_y(f) = \#_y(\mathcal{N}) = \deg\left[\frac{f(x') - y}{|f(x') - y|}\right]$$

and especially with $y = 0$ and $|f(x')| = 1$

$$\#_0(f) = \#_0(\mathcal{N}) = \deg f,$$

where $\deg f$ denotes the mapping degree of f.

Now, suppose $\Phi(x,\theta)$ to be a symbol of a Fredholmean singular integral operator, i.e.

$$\inf_{x,\theta} |\Phi(x,\theta)\overline{\Phi(x,\theta)}| > 0$$

then $\sqrt{\Phi(x,\theta)\overline{\Phi(x,\theta)}}$ is well-defined. Therefore, we set

$$\Psi(x,\theta) = \Psi_R(x,\theta) + i\Psi_I(x,\theta) := \frac{\Phi(x,\theta)}{\sqrt{\Phi(x,\theta)\overline{\Phi(x,\theta)}}}$$

thus $\Psi(x,\theta)$ fulfills

$$\Psi(x,\theta)\overline{\Psi(x,\theta)} = 1 \text{ which is equivalent to } \begin{cases} |\Psi_R(x,\theta)|^2 - |\Psi_I(x,\theta)|^2 = 1 \\ 2\langle\Psi_R(x,\theta),\, \Psi_I(x,\theta)\rangle = 0. \end{cases}$$

Using the homotopy

$$\Psi(x,\theta,t) := \sqrt{1 + (1-t)^2|\Psi_I(x,\theta)|^2}\,\frac{\Psi_R(x,\theta)}{|\Psi_R(x,\theta)|} + i(1-t)\Psi_I(x,\theta)$$

we see that

$$\Psi(x,\theta) \sim \frac{\Psi_R(x,\theta)}{|\Psi_R(x,\theta)|} =: \chi(x,\theta)$$

This last quaternionic-valued function fullfills

$$\chi(x,\theta)\overline{\chi(x,\theta)} = 1, \quad \overline{\mathbb{R}^2} \times S^1 \to S^3. \tag{13}$$

Due to a famous theorem of Hopf such mappings classify homotopy classes. We have to prove that (13) is related to the index of the singular integral operator with symbol $\Phi(x,\theta)$. We need the following lemma:

LEMMA 5. *(see* **[MP]***) Let ϕ be an arbitrary element of a topological group G and denote by $l_1(\phi)$ a homomorphism of G into the group of integers such that there is an element ϕ_1 with $l_1(\phi_1) = 1$. Furthermore, assume that there exists a homomorphism $l_0(\phi)$ of G into the group of integers with the property:*
If for some $\phi^* \in G$ the equation $l_1(\phi^*) = 0$ holds then $l_0(\phi^*) = 0$.
Under these assmptions we have

$$l_0(\phi) = \mu l_1(\phi) \text{ for any } \phi \in G, \text{ where } \mu \text{ is an integer.} \tag{14}$$

PROOF. Put $\mu = l_0(\phi_1)$ and $l_1(\phi) = r$ for an arbitrary but fixed element $\phi \in G$.

$$\text{Obviously,}\quad l_1(\phi\phi_1^{-r}) = l_1(\phi) - rl_1(\phi_1) = r - r = 0$$

$$\text{and hence}\quad l_0(\phi\phi_1^{-r}) = l_0(\phi) - rl_0(\phi_1) = l_0(\phi) - \mu l_1(\phi) = 0$$

which is equivalent to (14).

□

Adapting an idea from ([**MP**], page 376 f), we want to show that l_1 is the mapping degree of (13) and l_0 is the index of the singular integral operator with symbol χ, where χ is an element of the Spoin-group of $\mathbb{H}$, i.e. (13) is fulfilled. The mapping degree of (13) is also called rotation of the vector $\chi \in \mathbb{R}^4$ due to Krasnoselski or also index of the vector field $\chi \in \mathbb{R}^4$. We want to speak of the rotation of the quaternion χ meaning the mapping degree of (13).

Let $K(x,\theta) \in \text{Spoin}(\mathbb{H})$ be a quaternion with rotation ν. We choose an arbitrary quaternion $L(x,\theta) \in \text{Spoin}(\mathbb{H})$ and consider

$$K(x,\theta)^n = L(x,\theta),\ n \in \mathbb{N}, \tag{15}$$

we represent L in a geometric way, i.e.

$$L(x,\theta) = \cos(\alpha(x,\theta)) + \Lambda(x,\theta)\sin(\alpha(x,\theta)),$$

where α is the argument and Λ the axis of L. Then (15) is equivalent to n single equations

$$K(x,\theta) = \cos\left(\frac{\alpha + 2h\pi}{n}\right) + \Lambda \sin\left(\frac{\alpha + 2h\pi}{n}\right),\ h = 0, 1, \ldots, n-1. \tag{16}$$

Let the h-th equation of (16) have a_h and b_h solutions, resp., for which the Jacobian determinant of the mapping is positive or negative,resp., at a certain point. Then $a_h - b_h = \nu$ and (15) has $\sum_{h=0}^{n-1} a_h,\ \sum_{h=0}^{n-1} b_h$ solutions with positve or negative Jacobian determinant.
Therefore, the rotation of the quaternion $K(x,\theta)^n$ is $\sum_{h=0}^{n-1}(a_h - b_h) = n\nu$.

For $n = 0$ our statement is obviously true. Now, let $n = -m,\ m > 0,\ m \in \mathbb{N}$. Then (16) is of the form

$$K(x,\theta) = \cos\left(\frac{-\alpha + 2h\pi}{n}\right) + \Lambda \sin\left(\frac{-\alpha + 2h\pi}{n}\right),\ h = 0, 1, \ldots, n-1.$$

A sign chancing in α causes a sign changing in the Jacobian determinant. Hence the rotation of the quaternion $K^{(-1)}(x,\theta)$ is $-\nu$.

Now, if $K_1(x,\theta), K_2(x,\theta) \in \text{Spoin}(\mathbb{H})$ are two quaternions with rotations ν_1 an ν_2, respectivly, and $L(x,\theta) \in \text{Spoin}(\mathbb{H})$ an arbitrary quaternion with rotation 1 then K_1 and L^{ν_1} have the same rotation as well as K_2 and L^{ν_2}. Thus K_1 and L^{ν_1} as well as K_2 and L^{ν_2} are homotopic by the well-known Hopf theorem ([**AH**]). Hence, K_1K_2 is homotopic to $L^{\nu_1+\nu_2}$, and since homotopic quaternions have the same rotation we get that K_1K_2 has rotation $\nu_1 + \nu_2$. Consequently, the rotation of a quaternion provides a homorphism of the topological group Spoin($\mathbb{H}$) into the group of integers.

Let U be an element of Spoin($\mathbb{H}$) with rotation zero. Then U is homotopic to $Q = 1e_0$. Hence, the index of the singular integral operator associated to the symbol U is zero. This means if the rotation of U is zero then also the index is zero and applying Lemma 5 we get

THEOREM 3. *Let be $\chi(x,\theta)$ an element of the ring $\mathcal{R}$ defined in Lemma 3 and $\chi(x,\theta) \in$ Spoin($\mathbb{H}$). Then, the index of the singular integral operator with symbol $\chi(x,\theta)$ denoted by $\deg\chi$ is equal μ-times the mapping degree of $\overline{\chi(x,\theta)}\chi(x,\theta) = 1$ denoted by $\deg\chi$, i.e.*

$$\operatorname{ind}\chi = \mu\deg\chi = \mu\,\#_0(\mathcal{N}), \ \textit{where } \mu \textit{ is an integer and } \mathcal{N} : \overline{\chi}\chi = 1.$$

Coming back to the singular integral operator with symbol $\Phi \in \mathcal{R}$ we have

$$\operatorname{ind}\Phi = \operatorname{ind}\sqrt{\overline{\Phi}\Phi} + \operatorname{ind}\chi = \operatorname{ind}\sqrt{\overline{\Phi}\Phi} + \mu\deg\chi = \mu\deg\chi,$$

if $\sqrt{\overline{\Phi}\Phi}$ is smooth enough ([**MP**], page 319, Theorem 3.1).

References

[AH] Alexandroff, P., Hopf, H.: *Topologie I,* Springer Verlag, Berlin, 1935.

[Be1] Bernstein, S.: *On the left linear Riemann problem in Clifford analysis,* Bull. Belg. Math. Soc., Simon Stevin, **2**, 1996, 557-576.

[Be2] Bernstein, S.: *Linear and Nonlinear Riemann problems in Clifford Analysis,* In: Clifford Algebras and Their Application in Mathematical Physics, Aachen 1996 (eds.: V. Dietrich, K. Habetha and G. Jank), Kluwer Academic Publ., Dordrecht, Boston, London, 1998.

[Be3] Bernstein, S.: *Monotonicity principles for singular integral equations in Clifford analysis,* Preprint UofA-R-165 , University of Arkansas, Department of Mathematics, 1998.

[BDS] Brackx, F., Delanghe, R. and Sommen, F.: *Clifford Analysis* Research Notes in Mathematics 76, Pitman Adv. Publ. Program, 1982.

[DSS] Delanghe, R., Sommen, F. and Souček, V.: *Clifford algebra and Spinor-valued functions,* Kluwer Academic Publ., Dordrecht, Boston, London, 1992.

[GM] Gilbert, J.E., Murray, M.A.M.: *Clifford Algebras and Dirac Operators in Harmonic Analysis,* Cambridge studies in advanced mathematics vol. 26, Cambridge University Press, Cambridge, 1991.

[GS1] Gürlebeck, K., Sprößig, W.: *Quaternionic Analysis and Elliptic Boundary Value Problems* , Birkhäuser Verlag, Basel, 1990.

[GS2] Gürlebeck, K., Sprößig, W.: *Quaternionic and Clifford calculus for Engineers and Physicists,* Wiley & Sons Publ., 1997.

[Ha] Habetha, K.: *Eine Definition des Kroneckerindexes in $\mathbb{R}^{n+1}$ mit Hilfe der Cliffordanalysis,* Zeitschrift für Analysis und ihre Anwendungen, 5(2), 1986, 133-137.

[HS] Hestenes, D., Sobczyk, G.: *Clifford algebra to Geometric Calculus,* D. Reidel Publ. Company, Dordrecht, Holland, 1985.

[KS] Kravchenko, V.V., Shapiro M.V.: *Integral representations for spatial models of mathematical physics,* Pitman Research Notes in Mathematics Series 351.

[MLS] McIntosh, A., Li, C. and Semmes, S.: *Convolution singular integrals on Lipschitz surfaces,* Journal of the American Mathematical Society, 5: 455-481.

[MLQ] McIntosh, A., Li, C. and Qian, T.: *Clifford algebras, Fourier transforms and singular convolution operators on Lipschitz surfaces,* Revista Mathemática Iberoamericana 10: 665-721.

[MP] Mikhlin, S.G., Prößdorf, S.:*Singular Integral Operators,* Springer Verlag, Berlin, Heidelberg, New York, Tokyo, 1986.

[Pr] Prößdorf, S.:*Einige Klassen singulärer Gleichungen,* Akademie Verlag, Berlin, 1974.

[Sh1] Shapiro, M.V.:*On some boundary value problem for functions with values in in Clifford algebras,* Mat. Vestnik, Beograd 40 (1988), 321-326.

[Sh2] Shapiro, M.V.:*On analogoues of the Riemann boundary value problem for a class of hyperholomorphic functions,* In: Integral equations and boundary value problems, World Scientific, 1991, 184-188.

[ShV] Shapiro, M.V., Vasilevski, N.L. *Quaternionic ψ-holomorphic functions, singular integral operators and boundary value problems*, I, II, Complex Variables, Theory and Applications, **27**, 1995, 14-46 and 67-96.

[Su] Sudbery, A.: *Quaternionic Analysis,* Math. Proc. Cambridge Phil. Soc. **86**, (1979), 199-225.

[vW] von Wolfersdorf, L.: *Monotonicity Methods for Nonlinear Singular Integral and Integro-Differential Equations*, ZAMM **63**, 249-259(1983).

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