UNIQUENESS OF ∞ HARMONIC FUNCTIONS ON THE GRUSHIN PLANE

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ABSTRACT. This paper examines infinite harmonic functions in the viscosity sense on the Grushin plane. Existence of infinite harmonic functions in the viscosity sense is proved following the scheme of Bhatthacharya, DiBenedetto, and Manfredi (1989). Uniqueness of infinite harmonic functions is proved using an extension of Jensen's proof (1993). Both the existence and uniqueness proofs utilize the concept of subelliptic jets. By establishing a natural relationship between Euclidean and subelliptic jets, the viscosity solution technology of Crandall, Ishii, and Lions (1992) can be used .

1. PROPERTIES OF THE GRUSHIN PLANE

Consider the vector fields

$$X_1 = \frac{\partial}{\partial x} \quad \text{and} \quad X_2 = x\frac{\partial}{\partial y}$$

on $\mathbb{R}^2$. They form a basis at every point with $x \neq 0$. Applying the Lie Bracket yields

$$[X_1, X_2] = \frac{\partial}{\partial y} \equiv X_3.$$

Thus, at all points, $\{X_1, X_2, X_3\}$ generates $\mathbb{R}^2$. Endow $\mathbb{R}^2$ with an inner product (singular at points with $x = 0$) so that X_1 and X_2 are orthonormal. This environment, denoted g_2, is the tangent space to the Grushin plane, denoted G_2. The natural metric on G_2 is the Carnot-Carathéodory distance, which is defined for the points p and q as follows:

$$d_C(p,q) = \inf_{\Gamma} \int_0^1 \|\gamma'(t)\| dt$$

where the set Γ is the set of all curves γ such that $\gamma(0) = p, \gamma(1) = q$ and $\gamma'(t)$ is in span$\{X_1(\gamma(t)), X_2(\gamma(t))\}$. By Chow's theorem (see, for example, [BR]) any two points can be connected by such a curve, which means $d_C(p,q)$ is an honest metric. Define a Carnot-Carathéodory ball of radius r centered at a point p_0 by

$$B = B(p_0, r) = \{p \in G_2 : d_C(p, p_0) < r\}.$$

Using the procedure from [BR], it is seen that for $p_0 = (x_0, y_0)$ and $p = (x, y)$,

$$d_C(p_0, p) \sim |x| + |y|^{\frac{1}{2}} \quad \text{when} \quad x_0 = 0 \tag{1.1}$$

$$d_C(p_0, p) \sim |x| + |y| \quad \text{when} \quad x_0 \neq 0 \tag{1.2}$$

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Having established the basic structure on G_2, our attention turns to differentiation and calculus. Given a smooth function f on G_2, and a multi-index $I = (i_1, i_2, i_3)$, the derivative $X^I f$ is defined by

$$X^I f = X_1^{i_1} X_2^{i_2} X_3^{i_3} f.$$

The function f is C^k if $X^I f$ is continuous for all multi-indices I such that

$$d(I) \equiv i_1 + i_2 + 2i_3 \leq k. \tag{1.3}$$

In light of the Carnot-Carathéodory metric, the important first and second order derivatives that we will consider are given by

$$\nabla_0 f(p) = (X_1 f(p), X_2 f(p))$$

and

$$(D^2 f(p))^\star = \begin{pmatrix} X_1X_1f(p) & \frac{1}{2}(X_1X_2f(p) + X_2X_1f(p)) \\ \frac{1}{2}(X_1X_2f(p) + X_2X_1f(p)) & X_2X_2f(p) \end{pmatrix}.$$

Using these derivatives, define the horizontal p-Laplacian (for $p > 2$) by

$$\Delta_{0,p} f = \operatorname{div}(\|\nabla_0 f\|^{p-2} \nabla_0 f).$$

Formally taking the limit as p goes to infinity results in the horizontal infinite Laplacian which is defined by

$$\begin{aligned} \Delta_{0,\infty} f &= \sum_{i,j=1}^{2} X_i f X_j f X_i X_j f \\ &= \langle (D^2 f)^\star \nabla_0 f, \nabla_0 f \rangle \end{aligned}$$

For a more complete discussion of the p-Laplacian and infinite Laplacian, see [B]. It should also be noted that for any open set $\mathcal{O} \subset G_2$, the function f is in the horizontal Sobolev space $HW^{1,q}(\mathcal{O})$ if f, $X_1 f$ and $X_2 f$ are in L^q. Replacing L^q by L^q_{loc}, the space $HW^{1,q}_{loc}$ is defined similarly. The space $HW^{1,q}_0$ is the closure in $HW^{1,q}$ of smooth functions with compact support.

Within this environment, we will define infinite harmonic functions in the viscosity sense. The main result of this paper is that on a given domain with boundary data, there is a unique infinte harmonic function in the viscosity sense satisfying those boundary conditions.

2. TAYLOR POLYNOMIALS

In order to proceed, our attention must turn to Taylor polynomials. There are two forms of the Taylor polynomial on the Grushin plane, depending on the location of the base point. The following proposition formalizes this fact.

Proposition 2.1. *Let $f : G_2 \mapsto \mathbb{R}$ be a C^2 function. Let p_0 be denoted by (x_0, y_0). Then, if $x_0 = 0$,*

$$\begin{aligned} f(p) &= f(p_0) + xX_1 f(p_0) + (y - y_0)X_3 f(p_0) \\ &+ \frac{1}{2} x^2 X_1^2 f(p_0) + o(d_C(p_0, p)^2). \end{aligned}$$

If $x_0 \neq 0$,

$$\begin{aligned} f(p) &= f(p_0) + (x-x_0)X_1 f(p_0) + \frac{1}{x_0}(y-y_0)X_2 f(p_0) \\ &+ \frac{1}{2}(x-x_0)^2 X_1^2 f(p_0) + \frac{1}{2x_0}(y-y_0)^2 X_2^2 f(p_0) \\ &+ (x-x_0)(y-y_0)\Big(\frac{1}{x_0}X_1X_2 f(p_0) - \frac{1}{x_0^2}X_2 f(p_0)\Big) + o(d_C(p_0,p)^2). \end{aligned}$$

Proof: Case 1 : $x_0 \neq 0$.
Define the polynomial $P(p)$ by

$$\begin{aligned} P(p) &= f(p_0) + (x-x_0)X_1 f(p_0) + \frac{1}{x_0}(y-y_0)X_2 f(p_0) \\ &+ \frac{1}{2}(x-x_0)^2 X_1^2 f(p_0) + \frac{1}{2x_0^2}(y-y_0)^2 X_2^2 f(p_0) \\ &+ (x-x_0)(y-y_0)\Big(\frac{1}{x_0}X_1X_2 f(p_0) - \frac{1}{x_0^2}X_2 f(p_0)\Big) \end{aligned}$$

Then computation shows the following equations hold:

$$\begin{aligned} X_1P(p) &= X_1 f(p_0) + (x-x_0)X_1^2 f(p_0) + (y-y_0)\Big(\frac{1}{x_0}X_1X_2 f(p_0) - \frac{1}{x_0^2}X_2 f(p_0)\Big), \\ X_2P(p) &= \frac{x}{x_0}X_2 f(p_0) + \frac{x}{x_0^2}(y-y_0)X_2^2 f(p_0) + x(x-x_0)\Big(\frac{1}{x_0}X_1X_2 f(p_0) - \frac{1}{x_0^2}X_2 f(p_0)\Big), \\ X_1X_2P(p) &= \frac{1}{x_0}X_2 f(p_0) + \frac{1}{x_0^2}(y-y_0)X_2^2 f(p_0) + (2x-x_0)\Big(\frac{1}{x_0}X_1X_2 f(p_0) - \frac{1}{x_0^2}X_2 f(p_0)\Big), \\ X_2X_1P(p) &= x\Big(\frac{1}{x_0}X_1X_2 f(p_0) - \frac{1}{x_0^2}X_2 f(p_0)\Big), \\ X_1X_1P(p) &= X_1^2 f(p_0), \qquad \text{and} \\ X_2X_2P(p) &= \frac{x^2}{x_0^2}X_2^2 f(p_0). \end{aligned}$$

Evaluation at p_0 and recalling the relations

$$\begin{aligned} X_3 &= [X_1, X_2] \\ X_2 &= x_0 X_3 \end{aligned}$$

gives $X^I P(p_0) = X^I f(p_0)$ for $d(I) \leq 2$. By Theorem 4.10 in [BR], $f(p) - P(p)$ is $O(d_C(p_0,p)^3)$ and so it is $o(d_C(p_0,p)^2)$. In addition, note that by Equation (1.2) $y - y_0$ is $O(d_C(p_0,p))$.

Case 2 : $x_0 = 0$.
The proof is similar to the above case, except now $(y-y_0)$ is $O(d_C(p_0,p)^2)$ and X_2 is the zero vector.

□

In order to emphasize symmetry, so that the subelliptic jets may be defined in a natural way, we will rewrite the polynomial for the case $x_0 = 0$ as

$$\begin{aligned} f(p) &= f(p_0) + xX_1f(p_0) + 2(y-y_0)\frac{1}{2}(X_1X_2f(p_0) + X_2X_1f(p_0)) \\ &+ \frac{1}{2}x^2X_1^2f(p_0) + o(d_C(p_0,p)^2). \end{aligned}$$

3. SUBELLIPTIC JETS.

Let S^m be the set of all real $m \times m$ symmetric matrices. Let $\eta \in g_2$ and $X \in S^2$ be given by

$$\eta = \eta_1 X_1 + \eta_2 X_2 \quad \text{and} \quad X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix}.$$

Given a function $u : G_2 \mapsto \mathbb{R}$, consider the following inequalities:

$$\begin{aligned} u(p) &\leq u(p_0) + x\eta_1 + 2(y-y_0)X_{12} + \frac{1}{2}x^2X_{11} \\ &\quad + o(d_C(p_0,p)^2) \text{ as } p \to p_0 \text{ when } x_0 = 0, \qquad (3.1) \\ u(p) &\leq u(p_0) + (x-x_0)\eta_1 + \frac{1}{x_0}(y-y_0)\eta_2 + \frac{1}{2}(x-x_0)^2X_{11} \\ &\quad + \frac{1}{2x_0}(y-y_0)^2X_{22} + (x-x_0)(y-y_0)\left(\frac{1}{x_0}X_{12} - \frac{1}{x_0^2}\eta_2\right) \\ &\quad + o(d_C(p_0,p)^2) \text{ as } p \to p_0 \text{ when } x_0 \neq 0. \qquad (3.2) \end{aligned}$$

Given an open set $\mathcal{O} \subset G_2$ and a function $u : \mathcal{O} \mapsto \mathbb{R}$, define the second order superjet of u at p_0, denoted $J^{2,+}u(p_0)$ by the following:

$$(\eta, X) \in J^{2,+}_{\mathcal{O}}u(p_0) \quad \Leftrightarrow \quad p, p_0 \in \mathcal{O} \text{ and (3.1) holds} \quad (x_0 = 0)$$

or

$$(\eta, X) \in J^{2,+}_{\mathcal{O}}u(p_0) \quad \Leftrightarrow \quad p, p_0 \in \mathcal{O} \text{ and (3.2) holds} \quad (x_0 \neq 0)$$

The second order subjet of u at p_0, denoted $J^{2,-}u(p_0)$, is defined by

$$J^{2,-}_{\mathcal{O}}u(p_0) = -J^{2,+}_{\mathcal{O}}(-u)(p_0).$$

Following [CIL], we define the closure of a jet by

$$\begin{aligned} \overline{J}^{2,+}u(p_0) &= \{(\eta, X) : \exists (p_n, \eta_n, X_n) \text{ so that } (\eta_n, X_n) \in J^{2,+}u(p_n) \\ &\qquad \text{and } (p_n, u(p_n), \eta_n, X_n) \to (p, u(p), \eta, X)\} \end{aligned}$$

The continuous function u is infinite harmonic in the viscosity sense if $\langle X\eta, \eta\rangle \leq 0$ for all $(\eta, X) \in J^{2,+}u$ and $\langle X\eta, \eta\rangle \geq 0$ for all $(\eta, X) \in J^{2,-}u$.

Having formally defined the concept of subelliptic jet on the Grushin plane, the following proposition characterizes the jets in terms of test functions that touch from above or below. This proposition and proof is an extension of Crandall [C].

Proposition 3.1. *Let u, h, and $\mathcal{O}$ be as above. Define the set*

$$K^{u,p_0} = \{(\nabla_0\phi(p_0), (D^2\phi(p_0))^\star) : u - \phi \ \text{ has a local max at } p_0.\}$$

Then, we have the equality

$$J^{2,+}_{\mathcal{O}}u(p_0) = K^{u,p_0}.$$

Proof: Let p_0 be the local maximum of $u - \phi$. Then, for p near p_0,

$$u(p) - \phi(p) \leq u(p_0) - \phi(p_0)$$

and so

$$u(p) \leq u(p_0) + \phi(p) - \phi(p_0).$$

Then, Proposition 2.1 yields

$$K^{u,p_0} \subset J^{2,+}_{\mathcal{O}} u(p).$$

In order to show the reverse inclusion, a function ϕ with a strict maximum at p_0 that has the appropriate derivatives will be constructed. Define the function $a : G_2 \mapsto \mathbb{R}$ by

$$a(p) = (x^4 + y^4)$$

for $p = (x, y)$. This function is C^2 and satisfies $a(p_0 - p) = O(d_C(p_0, p)^4)$. We will first consider the case when $x_0 = 0$. Then, given the pair $(\eta, X) \in J^{2,+}_{\mathcal{O}} u(p_0)$, let

$$z(r) = \sup \left\{ \left(u(p) - u(p_0) - x\eta_1 - 2(y - y_0)X_{12} - \frac{1}{2}X_{11} \right)^+ \right\}$$

where the sup is taken over all $p \in \mathcal{O}$ such that $a(p - p_0) \leq r$.

Proceed as in [B] to construct a C^2 function $\zeta : G_2 \mapsto \mathbb{R}$ so that

$$X^I(\zeta(p - p_0))(p_0) = 0$$

for all multi-indices I so that $d(I) \leq 2$. Define the function $\phi : G_2 \mapsto \mathbb{R}$ by

$$\phi(p) = \zeta(p - p_0) + a(p - p_0) + x\eta_1 + 2(y - y_0)X_{12} + \frac{1}{2}x^2 X_{11}.$$

With this definition, $\phi(p_0) = 0$ and so,

$$\begin{aligned} u(p) - \phi(p) - u(p_0) + \phi(p_0) + s &= u(p) - \zeta(p - p_0) - a(p - p_0) - x\eta_1 \\ &- 2(y - y_0)X_{12} - \frac{1}{2}x^2 X_{11} - u(p_0) + s \end{aligned}$$

By the construction of ζ, this gives

$$u(p) - \phi(p) - u(p_0) + \phi(p_0) + s \leq 0$$

in the region $s \leq a(p - p_0)$. Thus, $u - \phi$ has a strict local maximum at p_0.

In addition, computation of the derivatives gives:

$$\begin{aligned} X_1\phi(p_0) &= \eta_1, \\ X_1X_1\phi(p_0) &= X_{11}, \\ X_1X_2\phi(p_0) &= X_{12}, \\ \text{and} \qquad X_2X_1\phi(p_0) &= 0. \end{aligned}$$

From which it follows that ϕ has the desired properties and so

$$J^{2,+}_{\mathcal{O}} u(p) \subset K^{u,p_0}.$$

The case where $x_0 \neq 0$ is similar and omitted.

□

Due to the subelliptic structure of the Grushin plane on the y-axis, the Crandall, Ishii, Lions maximum principle [CIL] is not readily available. This next lemma shows explicitly how any traditional Euclidean superjet gives rise to a subelliptic superjet so that the [CIL] machinery may be employed.

Lemma 3.2. *Let the points $p, p_0 \in \mathbb{R}^2$ be denoted by $p = (x, y)$ and $p_0 = (x_0, y_0)$. Let $\eta \in \mathbb{R}^2$ and $X \in S^2$. Also, let $\langle \cdot, \cdot \rangle_E$ denote the Euclidean inner product in $\mathbb{R}^2$. Then, define the standard Euclidean superjet, denoted $J_\sharp^{2,+}$, by*

$$\begin{aligned} J_\sharp^{2,+}u(p_0) &= \{(\eta, X) : u(p) \leq u(p_0) + \langle \eta, p - p_0 \rangle_E \\ &+ \frac{1}{2}\langle X(p - p_0), p - p_0 \rangle_E + o(\langle p - p_0, p - p_0 \rangle_E) \text{ as } p \to p_0.\} \end{aligned} \tag{3.3}$$

If η and X are defined by

$$\eta = (\eta_1, \eta_2) \qquad X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix},$$

define the g_2 vector

$$\tilde{\eta} = \eta_1 X_1 + x_0 \eta_2 X_2$$

and the symmetric matrix Y by

$$\begin{pmatrix} X_{11} & x_0 X_{12} + \eta_2 \\ x_0 X_{12} + \eta_2 & x_0^2 X_{22} \end{pmatrix}.$$

Then, given $(\eta, X) \in \overline{J}_\sharp^{2,+}u(p_0)$, $(\tilde{\eta}, Y) \in \overline{J}^{2,+}u(p_0)$.

Proof:

Case 1: $(\eta, X) \in J_\sharp^{2,+}u(p_0)$. By [CIL], there is a function ϕ such that $u - \phi$ has a local maximum at p_0 and

$$(\nabla \phi(p_0), D^2\phi(p_0)) = (\eta, X).$$

The result follows from the previous lemma, Equations (1.1) and (1.2) and by converting $(\nabla \phi(p_0), D^2\phi(p_0))$ into $(\nabla_0 \phi(p_0), (D^2\phi(p_0))^*)$.

Case 2: $(\eta, X) \in \overline{J}_\sharp^{2,+}u(p_0)$.

Given $(\eta, X) \in \overline{J}_\sharp^{2,+}u(p_0)$, there is a sequence $\{p_n, \eta_n, X_n\} \in B \times g_2 \times S^2$ so that $(\eta_n, X_n) \in J_\sharp^{2,+}u(p_n)$ and $\{p_n, u(p_n), \eta_n, X_n\} \to (p_0, u(p_0), \eta, X)$. Now, (η_n, X_n) can be identified with $(\tilde{\eta_n}, Y_n) \in J^{2,+}u(p_n)$. By construction, $\tilde{\eta_n} \to \tilde{\eta}$ and $Y_n \to Y$. Thus, $(p_n, u(p_n), \tilde{\eta_n}, Y_n) \to (p_0, u(p_0), \tilde{\eta}, Y)$ and so we have $(\tilde{\eta}, Y) \in \overline{J}^{2,+}u(p_0)$.

□

4. EXISTENCE-UNIQUENESS OF INFINITE HARMONIC FUNCTIONS IN THE VISCOSITY SENSE.

The existence of infinite harmonic functions in the viscosity sense follows [B] and is omitted. Our attention then turns to proving uniqueness. However, before proceeding with the main theorem about uniqueness, a technical lemma from [JLM] which gives a function that approximates the identity and has useful properties is given without proof.

Lemma 4.1. *Let $A > 1$ and $\alpha > 1$ be given. Then, the function $f : \mathbb{R} \mapsto \mathbb{R}$ given by*

$$f(t) = \frac{1}{\alpha} \log(1 + A(e^{\alpha t} - 1))$$

satisfies $f(0) = 0, f'(t) > 1$ and $f''(t) < 0$ for all $t \geq 0$. In addition, f is invertible and $0 < f(t) - t < \frac{A-1}{\alpha}$ as $A \to 1^+$.

Proceeding with the uniqueness argument, we will fix $\varepsilon > 0$. Define

$$\begin{array}{rrcl} & F^{\varepsilon}_{\infty}(\nabla_0 u, (D^2u)^\star) & = & \min\{\|\nabla_0 u\|^2 - \varepsilon^2, -\Delta_{0,\infty} u\}, \\ & H^{\varepsilon}_{\infty}(\nabla_0 u, (D^2u)^\star) & = & \max\{\varepsilon^2 - \|\nabla_0 u\|^2, -\Delta_{0,\infty} u\}, \\ \text{and} & F_{\infty}(\nabla_0 u, (D^2u)^\star) & = & -\Delta_{0,\infty} u. \end{array}$$

Note that a viscosity solution to $F_\infty = 0$ is a viscosity subsolution to $F^{\varepsilon}_{\infty} = 0$ and a viscosity supersolution to $H^{\varepsilon}_{\infty} = 0$. Also, a viscosity solution to $F^{\varepsilon}_{\infty} = 0$ is a viscosity supersolution to $F_\infty = 0$ and a viscosity solution to $H^{\varepsilon}_{\infty} = 0$ is a viscosity subsolution to $F_\infty = 0$.

Theorem 4.2. *Let u be a viscosity subsolution to $F^{\varepsilon}_{\infty} = 0$ in B. Let v be a viscosity supersolution to $F^{\varepsilon}_{\infty} = 0$ in B. Suppose u and v are continuous in $\overline{B}$. Then,*

$$\sup_{p\in\overline{B}}(u(p) - v(p)) = \sup_{p\in\partial B}(u(p) - v(p)).$$

Proof: Suppose not. Then,

$$\sup_{p\in\overline{B}}(u(p) - v(p)) > \sup_{p\in\partial B}(u(p) - v(p)). \tag{4.1}$$

Replace v by w with $\|v - w\|_{L^\infty(B)}$ small. Let $w = f(v)$ for A close to one, with f as in Lemma 4.1. Then, let $\sup_{p\in\overline{B}}(u(p) - w(p))$ occur at the (interior) point p_0. Let $\phi \in C^2(B)$ so that $\phi(p_0) = w(p_0)$ and $\phi(p) < w(p)$ for $p \neq p_0$. Set $\Phi = f^{-1}(\phi)$, that is, $\phi = f(\Phi)$. Proceeding as in [B], we set

$$\mu(p) = \min\{\varepsilon^2(f'(v(p))^2 - 1), -f''(v(p))f'(v(p))^2\varepsilon^4\},$$

and obtain

$$\min\{\nabla_0\Phi(p_0) - \varepsilon^2, -\Delta_{0,\infty}\Phi(p_0)\} \geq \mu(p_0) > 0.$$

Thus, w is a strict supersolution of $F^{\varepsilon}_{\infty} = 0$.

Next, denote the points p and s by $p = (x_1, y_1)$ and $s = (x_2, y_2)$ and let $(p_\tau, s_\tau) = ((x_1^\tau, y_1^\tau), (x_2^\tau, y_2^\tau))$ be the maximum point of

$$u(p) - v(s) - \frac{\tau}{2}(x_1 - x_2)^2$$

in $\overline{B} \times \overline{B}$. Proceeding as in [CIL], there are subsequences $p_{\tau_i} \to p_0$ and $s_{\tau_i} \to p_0$. In addition, there exists symmetric matrices X^τ and Y^τ such that

$$(\Upsilon, X^\tau) \in \overline{J}^{2,+}_{\sharp} u(p_\tau) \text{ and } (\Upsilon, Y^\tau) \in \overline{J}^{2,-}_{\sharp} w(s_\tau)$$

where

$$\Upsilon = \tau(x_1^\tau - x_2^\tau, 0)$$

and with the property that

$$\langle X^\tau \epsilon, \epsilon\rangle_E - \langle Y^\tau \chi, \chi\rangle_E \leq 3\tau\langle \epsilon - \chi, \epsilon - \chi\rangle_E \tag{4.2}$$

where $\langle \cdot, \cdot \rangle_E$ is the standard Euclidean inner product.

By Lemma 3.2, $(\Upsilon, X_u) \in \overline{J}^{2,+} u(p_\tau)$ and $(\Upsilon, Y_w) \in \overline{J}^{2,-} w(s_\tau)$ where

$$\begin{array}{rrcl} & X_u & = & \begin{pmatrix} X_{11} & x_1^\tau X_{12} \\ x_1^\tau X_{12} & (x_1^\tau)^2 X_{22} \end{pmatrix} \\ \text{and} & Y_w & = & \begin{pmatrix} Y_{11} & x_2^\tau Y_{12} \\ x_2^\tau Y_{12} & (x_2^\tau)^2 Y_{22} \end{pmatrix} \end{array}$$

Since u is a subsolution and w is a strict supersolution,

$$\mu(s_\tau) \leq \min\{|\Upsilon|^2 - \varepsilon^2, -\langle Y_w \Upsilon, \Upsilon\rangle\}$$

and

$$0 \geq \min\{|\Upsilon|^2 - \varepsilon^2, -\langle X_u \Upsilon, \Upsilon\rangle\}.$$

Subtracting the latter from the former yields

$$0 < \mu(s_\tau) \leq \max\{0, \langle X_u \Upsilon, \Upsilon\rangle - \langle Y_w \Upsilon, \Upsilon\rangle\} = 0$$

by Equation (4.2). The contradiction implies the original supposition is false. □

Uniqueness of infinite harmonic functions then follows as in [B].

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