Weak *C*-sets are not *C*-sets in Regular Semigroups*

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Let S be a regular semigroup, and E_S the set of idempotents of S. Let P be a subset of E_S such that $P \cap L \neq \emptyset$ and $P \cap R \neq \emptyset$ for every $\mathcal{L}$-class L and $\mathcal{R}$-class R of S (where $\mathcal{L}$ and $\mathcal{R}$ are Green's L- and R-relations respectively). If the pair (S, P) of S and P satisfies

(C1) $P^2 \subset E_S$,

(C2) $qPq \subset P \qquad \forall q \in P$,

then $S(P)$ is called *weakly $\mathcal{P}$-regular* and P *a weak C-set.* If (S, P) further satisfies

(C3) for any $x \in S$, there exists $x^* \in V(x)$ (where $V(x)$ denotes the set of all inverses of x) such that $xP^1x^*, x^*P^1x \subset P$ (where P^1 is the adjunction of 1 to P),

then we say that $S(P)$ is *$\mathcal{P}$-regular* and P *a C-set* in S. In this case, x^* above is called a $\mathcal{P}$-inverse of x. and the set of all $\mathcal{P}$-inverses of x is denoted by $V_P(x)$.

In the previous papers for [weakly] $\mathcal{P}$-regular semigroups, such as [2-8], we have seen that the [weak] C-set of a [weakly] $\mathcal{P}$-regular semigroup plays an important part in studies on [weakly] $\mathcal{P}$-regular semigroups. For a regular semigroup S, it is obvious that if there exists a C-set in S, then there exists a weak C-set in S. Conversely, is it true? In the present work, a negative answer is given.

A regular semigroup is called *a cryptogroup* if it is a band of groups(see[6]). The following result was obtained by Yamada (see [6]):

Lemma 1. *For a cryptogroup S, P is a C-set in S iff it is a weak C-set in S.*

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Theorem 2. *Let $S = M(G_1; I, J; Q)$ be a completely simple semigroup, where $G_1 = <a>$ is an infinite cyclic group with the unit element e, $I = J = \{1,2,3,4\}$, and*

$$Q = (p_{ji})_{4\times 4} = \begin{bmatrix} e & a & a^{-1} & e \\ a^{-1} & e & e & a \\ a & e & e & a^{-1} \\ e & a^{-1} & a & e \end{bmatrix}.$$

Then, S has only two C-sets $P_1 = \{(i,e,i) : i \in I\}$ and $P_2 = \{(i,e,5-i) : i \in I\}$.

Proof. It is easy to see that $p_{ji} = p_{ij}^{-1}, p_{j,5-i} = p_{5-j,i}$ for any $i, j \in I$, and $E_S = \{(i, p_{ij}, j) : i, j \in I\}$. Since S is a cryptogroup and by Lemma 1, it is sufficient to prove that S has only two weak C-sets P_1 and P_2. For every $\mathcal{L}$-class L and $\mathcal{R}$-class R of S, it is obvious that $P_1 \cap L \neq \emptyset, P_1 \cap R \neq \emptyset, P_2 \cap L \neq \emptyset$, and $P_2 \cap R \neq \emptyset$. Now let $(i,e,i), (j,e,j) \in P_1$ and $(i,e,5-i), (j,e,5-j) \in P_2$. Then

$$(i,e,i)(j,e,j) = (i, ep_{ij}e, j) = (i, p_{ij}, j) \in E_S,$$

$$(i,e,i)(j,e,j)(i,e,i) = (i, p_{ij}p_{ji}, i) = (i,e,i) \in P_1,$$

$$(i,e,5-i)(j,e,5-j) = (i, ep_{5-i,j}e, 5-j) = (i, p_{i,5-j}, 5-j) \in E_S,$$

and

$$(i,e,5-i)(j,e,5-j)(i,e,5-i) = (i, p_{i,5-j}p_{5-j,i}, 5-i) = (i,e,5-i) \in P_2.$$

Thus, $P_1^2 \subset E_S, q_1P_1q_1 \subset P_1$ for any $q_1 \in P_1$, and $P_2^2 \subset E_S$, $q_2P_2q_2 \subset P_2$ for any $q_2 \in P_2$. Therefore, both P_1 and P_2 are weak C-sets in S. Finally suppose that P is any weak C-set in S. We wish to prove that $P = P_1$ or P_2.

Suppose that $(1,a,2) \in P$. Since $(1,a,2)(2,e,3) = (1,a,3) \notin E_S, (1,a,2)(3,e,3) = (1,a,3) \notin E_S$, and $(1,a,2)(4,a,3) = (1,a^3,3) \notin E_S$, we have that

$$(i, p_{i3}, 3) \notin P, i = 2,3,4 \tag{1}$$

by (C1). Thus $(1,a^{-1},3) \in P$, because $P \cap L \neq \emptyset$ for every $\mathcal{L}$-class L. As

$$(1,a^{-1},3)(4,e,1) = (1,a^{-2},1) \notin E_S,$$

$$(1,a^{-1},3)(4,a^{-1},2) = (1,a^{-3},2) \notin E_S$$

and

$$(1,a^{-1},3)(4,e,4) = (1,a^{-2},4) \notin E_S,$$

we have that $(4, p_{4j}, j) \notin P, j = 1, 2, 4$ by (C1). Hence, $(4, a, 3) \in P$ follows from the fact that $P \cap R \neq \emptyset$ for every $\mathcal{R}$-class R, which is in contradiction with (1). Therefore,

$$(1, a, 2) \notin P. \tag{2}$$

Suppose that $(1, a^{-1}, 3) \in P$. Because

$$(1, a^{-1}, 3)(2, e, 2) = (1, a^{-1}, 2) \notin E_S,$$

$$(1, a^{-1}, 3)(3, e, 2) = (1, a^{-1}, 2) \notin E_S$$

and

$$(1, a^{-1}, 3)(4, a^{-1}, 2) = (1, a^{-3}, 2) \notin E_S,$$

we have that

$$(i, p_{i2}, 2) \notin P, i = 2, 3, 4$$

by (C1). Thus, $(1, a, 2) \in P$, which is in contradiction with (2). Therefore,

$$(1, a^{-1}, 3) \notin P. \tag{3}$$

Now, from (2) and (3) and because $P \cap R \neq \emptyset$ for every $\mathcal{R}$-class R, we get $(1, e, 1)$ or $(1, e, 4) \in P$. Notice that $(4, a^{-1}, 2)(1, e, 1) = (4, a^{-2}, 1) \notin E_S, (4, a^{-1}, 2)(1, e, 4) = (4, a^{-2}, 4) \notin E_S, (4, a, 3)(1, e, 1) = (4, a^2, 1) \notin E_S$, and $(4, a, 3)(1, e, 4) = (4, a^2, 4) \notin E_S$. Therefore, by (C1) we have

$$(4, a^{-1}, 2), (4, a, 3) \notin P. \tag{4}$$

Notice that Q is a symmetric matrix. It follows from the proof of (2), (3), and (4) that

$$(2, a^{-1}, 1), (3, a, 1), (2, a, 4), (3, a^{-1}, 4) \notin P. \tag{5}$$

Hence we have

$$P \subset P_1 \cup P_2$$

by using (2), (3), (4), and (5) above.

Also, because $P \cap L \neq \emptyset$ for every $\mathcal{L}$-class L, we have that $(1, e, 1)$ or $(1, e, 4) \in P$.

If $(1, e, 1) \in P$, then it follows from

$$(1, e, 1)(2, e, 3) = (1, a, 3) \notin E_S$$

and

$$(1, e, 1)(3, e, 2) = (1, a^{-1}, 2) \notin E_S$$

that

$$(2,e,3),(3,e,2) \notin P. \tag{6}$$

Now, from (6), we get

$$(2,e,2),(3,e,3) \in P. \tag{7}$$

From (7) and because

$$(2,e,2)(4,e,1) = (2,a,1) \notin E_S$$

and

$$(2,e,2)(1,e,4) = (2,a^{-1},4) \notin E_S,$$

we have

$$(4,e,1),(1,e,4) \notin P. \tag{8}$$

From (8), we get

$$(4,e,4) \in P. \tag{9}$$

Therefore we have $P = P_1$ by using (6), (7), (8), and (9). Similarly, if $(1,e,4) \in P$, we have $P = P_2$. This completes the proof.

Let $G_2 = \{f, b\}$ be a group with the unit element f, σ_1 and σ_2 two permutations of the set {1, 2, 3, 4}, where

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \quad \text{and} \quad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}.$$

Let $T = S \cup G_2$, where $S = M(G_1; I, J; Q)$. Now, define multiplication $\circ$ in T as follows:

$$x \circ y = \begin{cases} xy \text{ in } S & \text{if } x,y \in S \\ xy \text{ in } G_2 & \text{if } x,y \in G_2 \\ ((i)\sigma_1, g, j) & \text{if } x = b, y = (i,g,j) \in S \\ (i, g, (j)\sigma_2) & \text{if } x = (i,g,j) \in S, y = b \\ y & \text{if } x = f, y \in S \\ x & \text{if } x \in S, y = f. \end{cases}$$

Then we have

Theorem 3. *$(T, \circ)$ is a regular semigroup, and it has only two weak C-sets $Q_1 = \{f\} \cup P_1$ and $Q_2 = \{f\} \cup P_2$, but neither of them is a C-set in T.*

Proof. To show T is a semigroup, let $x, y, z \in T$.

If $x, y, z \in G_2$ or $x, y, z \in S$ or only one belongs to S among x, y, and z, it is easy to see that $(x \circ y) \circ z = x \circ (y \circ z)$.

Suppose that only one element belongs to G_2 among x, y, and z. If the element is f or $x = b$ or $z = b$, then it is clear that $(x \circ y) \circ z = x \circ (y \circ z)$. Now, let $x = (i, g, j) \in S, y = b$ and $z = (\lambda, h, \mu) \in S$. Then

$$(x \circ y) \circ z = (i, gp_{(j)\sigma_2,\lambda}h, \mu) \quad \text{and} \quad x \circ (y \circ z) = (i, gp_{j,(\lambda)\sigma_1}h, \mu).$$

Since

$$(p_{(j)\sigma_2,\lambda})_{4\times 4} = \begin{bmatrix} a^{-1} & e & e & a \\ e & a & a^{-1} & e \\ e & a^{-1} & a & e \\ a & e & e & a^{-1} \end{bmatrix} = (p_{j,(\lambda)\sigma_1})_{4\times 4},$$

we have that $(x \circ y) \circ z = x \circ (y \circ z)$ for any $x, z \in S$ and $y = b$.

From the argument above, it follows that $(T, \circ)$ is a semigroup. It is straightforward that T is a regular semigroup, and both Q_1 and Q_2 are weak C-sets in T. Now, suppose that Q_0 is any weak C-set in T. It is easy to see that $Q_0 \cap S$ is a weak C-set in S. thus, we have that $Q_0 \cap S = P_1$ or P_2 by Theorem 2. Since $Q_0 \cap L \neq \emptyset$ and $Q_0 \cap R \neq \emptyset$ for every $\mathcal{L}$-class L and $\mathcal{R}$-class R of T, we have that $f \in Q_0$. Therefore, $Q_0 = \{f\} \cup P_1$ or $\{f\} \cup P_2$, that is, Q_1 and Q_2 are only two weak C-sets in T. Notice that $V(b) = \{b\}$ in T, $bQ_1^1 b = Q_2$, and $bQ_2^1 b = Q_1$. Hence, by (C3) we have that neither of Q_1 and Q_2 is a C-set in T.

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