A list of open problems
presented at the Spring Lecture Series 2001
"Solutions of PDE's in periodic media"

1) Dmitry Khavinson (University of Arkansas) proposed the following problem: Let Γ be an analytic hypersurface in $\mathbb{R}^n$, $n \geq 3$, and let u be a solution to

$$\begin{cases} \Delta u = 1 \\ u = 0, \nabla u = 0 \text{ on } \Gamma \end{cases} \tag{1}$$

Conjecture *If u is entire then Γ is a plane.*

The case $n = 2$ has been solved by P. Davis using complex variables methods. In higher dimension there are some related results by Caffarelli, Karp and Shahgholian. Special cases have been studied by Khavinson and Shapiro.

2) John Mather (Princeton) proposed the following problem, related to work by Bangert and Burago-Kleiner-Ivanov: *Let M denote a compact, smooth (infinitely differentiable) Riemannian manifold. Does the stable norm have the Bangert-Burago-Kleiner-Ivanov differentiability property?* To be more precise, let $|| \cdot ||$ denote the stable norm, and $B \subset H_1(M, \mathbb{R})$ denote its unit ball. Let $h \in H_1(M, \mathbb{R}) \cap \partial B$ and let V_h denote the smallest subspace over $\mathbb{Q}$ of $H_1(M, \mathbb{Q})$ such that $h \in V_h \otimes_{\mathbb{Q}} \mathbb{R}$. The differentiablity property says that $(V_h \otimes_{\mathbb{Q}} \mathbb{R}) \cap B$ has only one supporting hyperplane in $V_h \otimes_{\mathbb{Q}} \mathbb{R}$. This property fails in high dimension if the metric is not smooth enough.

3) Luis Caffarelli (University of Texas) proposed the study of a large class of problems in homogenization theory, with the following underlying philosophy: *In every problem in classical mechanics that admits travelling wave solutions in a homogeneous media, there should be a "almost-travelling wave" solution in the non-homogeneous setting (in the homogenization limit).*

As a first example, Caffarelli recalled the study of flame propagation, studied by Berestycki, P.L. Lions and Nirenberg, where "Traveling flames" solutions of semilinear equations of the form $\Delta u = f(u)$, with $u = -1$ at ∞ were obtained in a homogeneous media. If the media is not homogeneous, one cannot expect the existence of such solutions, but in the homogenization limit it should be possible to construct almost-traveling wave solutions. A second example comes from the study of motion by mean curvature.

Caffarelli also pointed out the program (originally proposed by F. Otto) to study evolution equations via optimal transport.

4) Victor Bangert (Albert-Ludwigs-Universität Freiburg, Germany) proposed the following Liouville-type problem for minimal solutions of functionals of the form

$$\int F(x, u(x), \nabla u(x))dx \tag{2}$$

where $F(x, u, p)$ is periodic in $x \in \mathbb{R}^d$ and $u \in \mathbb{R}$ and satisfies convexity and growth conditions in $p \in \mathbb{R}^d$ as in [3]. The question is if every minimal solution $u : \mathbb{R}^d \to \mathbb{R}$ of (2) that is of linear growth is automatically "without self-intersections", i.e. if for all $k \in \mathbb{Z}^d, j \in \mathbb{Z}$, the function $x \to u(x-k) - u(x) + j$ does not change sign. In particular this would imply the existence of $\alpha = \alpha(u) \in \mathbb{R}^d$ such that $x \to u(x) - \alpha \cdot x$ is bounded. The answer is yes if $d = 1$; here one does not even need the hypothesis that u has linear growth, cf. [1], Proposition 7. It is also yes if $d \geq 1$ and $F_u \equiv 0$, cf. [3]. The problem was first stated in [2], sect. 8, which also contains some partial results in the general case.

[1] S. Aubry, P.Y. LeDaeron: *The discrete Frenkel-Kontorova model and its extensions I. Exact results for the ground-states*, Physica 8D (1983), 381-422.

[2] V. Bangert: *On minimal laminations of the torus*, Ann. Inst. H. Poincaré-Analyse non linéaire 6 (1989), 95-138.

[3] J. Moser, M. Struwe: *On a Liouville-type theorem for linear and nonlinear elliptic differential equations on a torus*, Bol. Soc. Brasil. Mat. 23 (1992), 1-20.

5) John Mather (Princeton) proposed the following question: Let G_n^r the group of C^r "compactly supported" diffeomorphisms of $\mathbb{R}^n$ (i.e. equal to the identity outside of a compact set). Let $G_n^{r,0}$ be the subgroup of diffeomorphisms isotopic to the identity.

Question: Is $G_n^{r,0}$ perfect (i.e. equal to its commutator subgroup)?

The answer is positive if $r \neq n+1$. The problem is open if $r = n+1$.

6) Dmitry Burago (PennState) proposed the following problem: Let M be a complete non-compact manifold, and Γ a group which acts on M co-compactly with respect to two metrics d_1 and d_2. Assume

$$\lim_{d(x,y)\to\infty} \frac{d_1(x,y)}{d_2(x,y)} = 1$$

Question: Is it true that

$$|d_1(x,y) - d_2(x,y)| \leq C?$$

The answer is positive in special cases (hyperbolic groups, or the Heisenberg group), and open in general. There are no known counter-examples. A good starting point would be the case where Γ is semi-hyperbolic.